Week 1: Functions and Equations

Goals:

- Review functions
- Introduce modelling using linear and quadratic functions
- Solving equations and systems

Suggested Textbook Readings: Chapter 2: §2.1 - §2.2, and Chapter 3: §3.1-§3.6

Practice Problems:

- §2.1: 3, 23, 31
- §3.1: 15, 19, 23, 27, 49, 65
- §3.2: 17, 21, 23
- §3.3: 31
- §3.4: 7, 11
- §3.6: 13, 15, 17, 21 (13th edition, the question is given below)
- Review Problems - 49, 51

§3.6 - 21

Business (a) Find the break-even points for company X, which sells all it produces, if the variable cost per unit is $3, fixed costs are $2, and \( y_{TR} = 5\sqrt{q} \), where \( q \) is the number of thousands of units of output produced.

(b) Graph the total revenue curve and the total cost curve in the same plane.

(c) Use your answer in (a) to report the quantity interval in which maximum profit occurs.
What is a Function?

To prepare for this topic, please read sections §2.1 and §2.2 in the textbook.

A function is a rule that assigns to each input number exactly one output number. The set of all input numbers to which the rule applies is called the domain of the function. The set of all output numbers is called the range.

A variable that represents input numbers for a function is called an independent variable. A variable that represents output numbers is called a dependent variable.

[Textbook, Section 2.1]

Let's keep this mathematical definition in mind as we think about the many ways in which we encounter functions.

Example 1: The following table is a supply schedule. It describes the relationship between the price $p$ per unit of a certain product and the quantity $q$ the producer will supply per week at that price.

<table>
<thead>
<tr>
<th>Price $p$ ($) per unit</th>
<th>Quantity supplied $q$ per week</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>11</td>
</tr>
<tr>
<td>600</td>
<td>14</td>
</tr>
<tr>
<td>700</td>
<td>17</td>
</tr>
<tr>
<td>800</td>
<td>20</td>
</tr>
</tbody>
</table>

The price $p$ is a function of the quantity $q$, i.e. $P = P(q)$

if $q = 14$, $P = P(14) = 600$

Example 2: The following graph shows the approximate value $V$ (in thousand dollars) of a car after being manufactured $t$ years.

The function $V = V(t)$ is represented by a graph

$t$: independent variable

$V$: dependent variable

We may find the value of function by estimating the height of a point on the graph.

For example:

if $t = 3$, $V(3) = 15$

if $t = 1$, $V = V(1) = 24$

if $t = 8$, $V = V(8) = 5$.

The domain is $0 \leq t \leq 10$

The range is $0 < V < 30$
Example 3: Suppose that the equation \( p = 100/q \) describes the relationship between the price \( p \) per unit of certain product and the number of units \( q \) of the product that customers will buy (that is, the demand) per week at the stated price. Such an equation is called a demand equation for the product.

\[ p = p(q) \text{ is a function defined by } p = \frac{100}{q} \]

\( q \) is the independent variable and \( p \) is the dependent variable.

The domain is \( q > 0 \) and the range is \( p > 0 \).

The value of the function at \( q \) can be calculated by \( p = \frac{100}{q} \)

- if \( q = 1 \), \( p = 100 \)
- if \( q = 5 \), \( p = 20 \).

A function may be described in several different ways. It may be specified

1. algebraically by a formula;
2. graphically;
3. numerically by a table;
4. verbally.

Special functions we will study:

- **Constant Functions**: for example, \( y = 2 \), \( h(x) = 9 \).
- **Polynomial Functions**: for example, \( y = x + 2 \), \( f(x) = -x^3 + 2x^2 - 1 \).
- **Rational Functions**: for example, \( y = \sqrt{x^2 - 3} \).
- **Absolute-Value Function**:

\[
y = |x| = \begin{cases} 
x, & \text{if } x \geq 0 \\
-x, & \text{if } x < 0
\end{cases}
\]
Mathematical Modeling

To mathematically model a situation means to represent it in mathematical terms (formulas or graphs). The particular representation used is called a mathematical model of the situation. Mathematical models do not always represent a situation perfectly or completely. Some represent a situation only approximately, whereas others represent only some aspects of the situation. If a situation is modelled properly, the mathematical model(s) can give us a lot of insight into the problem.

Example 4: (Problem 2.2 - 33) To encourage large group sales, a theater charges two rates. If your group is less than 12 people, each ticket costs $9.50. If your group is 12 or more, each ticket costs $8.75. Model the total cost for your group of $n$ people.

\[
\text{cost} = \# \text{ of people in a group } \times \text{ the ticket price}
\]

so the total cost \( C = C(n) \) if there are \( n \) people in your group is

\[
C = \begin{cases} 
8.75n, & \text{if } n > 12 \\
9.5n, & \text{if } n \leq 12
\end{cases}
\]

Example 5: Just before the class, you bought a cup of freshly poured coffee to keep you awake in Math 126 class. You would like to drink it as soon as you can. Pick some parameters corresponding to the coffee that change over time.

- **Temperature** \( T = T(t) \)
- **Amount** \( A = A(t) \)

\[\text{Temp} \quad T \uparrow_{100^\circ C} \quad 20\] \[\text{time} \quad t \]

\[\text{Amount} \quad A \uparrow_{\text{Full}} \quad \text{Empty}\] \[\text{time} \quad t\]
Example 6: The supply and demand is an economic model based on price, utility and quantity in a market. For each price level of a product, there is a corresponding quantity of that product that consumers will demand during some time period. In response to various prices, there is a corresponding quantity that producers are willing to supply to the market. It conclude that in a competitive market, price will function to equalize the quantity demanded by consumers and the quantity supplied by producers. Draw the demand and supply curves. [Textbook, page 125(12th), or 136(13th)]

- Demand

An increase in demand corresponds to a decrease in price so the demand curve falls from left to right.

- Supply

An increase in supply corresponds to an increase in price, so the supply curve raises from left to right.
Linear Functions

To prepare for this topic, please read sections §3.1 and §3.2 in the textbook.
Linear functions are among the simplest functions and are perhaps the most useful of all mathematical functions.

Example 7: The Supply schedule is given below

<table>
<thead>
<tr>
<th>Price p ($ per unit)</th>
<th>Quantity supplied q per week</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>8</td>
</tr>
<tr>
<td>500</td>
<td>11</td>
</tr>
<tr>
<td>600</td>
<td>14</td>
</tr>
<tr>
<td>700</td>
<td>17</td>
</tr>
<tr>
<td>800</td>
<td>20</td>
</tr>
</tbody>
</table>

1. Draw the supply curve by plotting points.

2. What can you find from above curve?
   - If we connect all points, it is a straight line.
   - The ratio of vertical change and horizontal change between any two adjacent points is the same.

\[ m = \frac{100 - 8}{3} \]

Definition

Let \((x_1, y_1)\) and \((x_2, y_2)\) be two different points on a nonvertical line. The slope of the line is

\[ m = \frac{y_2 - y_1}{x_2 - x_1} = \text{vertical change} \]
\[ \text{horizonal change} \]

[Textbook, Section 3.1]
3. Predict the price when the supply quantity is 23 if the ratio of changes in price \( p \) and supply quantity \( q \) remains the same.

We pick the point \((20, \ 800)\) to start with. Then

\[
p = 800 + (23 - 20) \cdot \frac{100}{3} = 900
\]

4. Write a general formula for the relation between the Price \( p \) and the supply quantity \( q \).

\[
\text{ratio or slope : } m \\
\text{point : } \ (q_0, \ p_0) \\
p = p_0 + (q - q_0) \cdot m
\]

**Definition**

The **point-slope form** of a line through \((x_1, y_1)\) with slope \( m \) is

\[
y = m(x - x_1) + y_1
\]

[Textbook, Section 3.1]

**Concept Question 1** If \( m = 0 \), then the line is

\[
A. \ an \ upward \ line \\
B. \ a \ downward \ line \\
C. \ a \ horizontal \ line \\
D. \ a \ vertical \ line
\]

*Note: Let \( L_1 \) and \( L_2 \) be two lines with slope \( m_1 \) and \( m_2 \) respectively.*

- \( L_1 \parallel L_2 \iff m_1 = m_2 \) or \( L_1 \) and \( L_2 \) are both vertical.
- \( L_1 \perp L_2 \iff m_1m_2 = -1 \)
A function $f$ is linear if and only if $f(x)$ can be written in the form $f(x) = ax + b$, where $a$ and $b$ are constants and $a \neq 0$.

**Example 8**: (Example 2 on page 126(12th), or page 137(13th)) Suppose the demand per week for a product is 100 units when the price is $58 per unit and 200 units at $51 each. Determine the demand function, i.e., the function of the price $p$ as the demand quantity $q$, assuming that it is linear.

Note that

- **it is a line** $p = p(q)$
- **two points on the line**
  $(q_1, p_1) = (100, 58)$
  $(q_2, p_2) = (200, 51)$

The slope of the line

$$m = \frac{p_2 - p_1}{q_2 - q_1} = \frac{51 - 58}{200 - 100} = -0.07$$

we use point-slope form

- **point** $(q_1, p_1) = (100, 58)$
- **slope** $m = -0.07$

then

$$p = p_1 + m(q - q_1)$$
$$p = 58 + (-0.07)(q - 100)$$

i.e.

$$p = -0.07q + 65$$

is the demand function.
**Example 9:** (December Exam 2009 - Problem 4:) At a price of $60 for a half-day trip, a white-water rafting company attracts 50 customers. As the price is increased the number of customers will decrease **linearly**, more precisely, every $5 **decrease** in price attracts an additional 10 customers. Assume the maintenance and equipment cost for each trip is $500, and the company has to pay $5 insurance for every customer on board. The company will not be interested in a price less than $20, and if the price is more than $85, there will be no customers.

(a) Let the price of the trip be $x$ dollars. How many customers $N = N(x)$ are expected to attend the trip?

*Note that:*

- $N = N(x)$ is a linear function.
- Points on the line:
  - $(x_1, N_1) = (60, 50)$
  - $(x_2, N_2) = (55, 60)$

So the slope is

$$m = \frac{N_2 - N_1}{x_2 - x_1} = \frac{60 - 50}{55 - 60} = -2$$

**pick a point**

$(x_1, N_1) = (60, 50)$

**use point-slope form**

$$N = 50 + (-2)(x - 60)$$

i.e.

$$N = -2x + 170$$

(b) If the price is $x$, find the revenue function $R$ and the cost function $C$. What is the profit function $P$?

*If the price is $x$, then there are $N = -2x + 170$ customers.*

So the total revenue is

$$R = xN = x(-2x + 170) = -2x^2 + 170x$$

The total cost

$$C = \text{Fixed cost} + \text{Insurance}$$

$$= 500 + 5N = 500 + 5(-2x + 170)$$

$$= 1350 - 10x$$

Then the profit is

$$P = \text{Revenue} - \text{Cost}$$

$$= (-2x^2 + 170x) - (1350 - 10x)$$

$$= -2x^2 + 180x - 1350$$
Systems of Linear Equations and Applications

To prepare for this topic, please read sections §3.4 - §3.6 in the textbook.

To solve systems of linear equations, we usually use elimination method (by addition or substitution).

**Example 10:** Solve the system. \[
\begin{align*}
3x - 4y &= 13 \quad (A) \\
3y + 2x &= 3 \quad (B)
\end{align*}
\]

we use substitution.

- Solving \( A \) for \( y \) in terms of \( x \)
  \[ y = \frac{3}{4} x - \frac{13}{4} \quad (D) \]
- Substituting the right side of \( D \) for \( y \) in \( B \)
  \[ 3 \left( \frac{3}{4} x - \frac{13}{4} \right) + 2x = 3 \]
  \[ x = 3 \]
- Use \( D \) to find \( y \)
  \[ y = -1 \]

so there is only one solution. \( x = 3 \) and \( y = -1 \).

Geometrically, the solution of a system is the intersection of the two lines given by the two equations.

\[ \text{Equation B} \]
\[ y = -\frac{3}{2} x + 1 \]
\[ (3y + 2x = 3) \]

\[ \text{Equation A} \]
\[ y = \frac{3}{2} x - \frac{13}{4} \]
\[ (3x - 4y = 13) \]
Supply and Demand

Supply and demand is an economic model based on price, utility and quantity in a market. A higher price generally results in a lower demand. However, as the price rises, suppliers will be more inclined to produce these items (as opposed to spending their time and money on other products), so supply will generally rise and will typically result in a reduction in price and vice-versa. In a competitive market, price will function to equalize the quantity demanded by consumers and the quantity supplied by producers, resulting in an economic equilibrium of price and quantity.

Example 11: You run a speciality footwear store, and must determine how much to charge for SoComfy brand shoes. Suppose that the demand function is

\[ p = -\frac{1}{18} q + 120 \]

and the supply function is

\[ p = \frac{1}{30} q + 40 \]

1. Draw the two functions on the same coordinate plane below.

![Graph](image)

2. Find the equilibrium price and equilibrium quantity.

\[
\begin{align*}
\left\{ & \quad p = -\frac{1}{18} q + 120 \\
& \quad p = \frac{1}{30} q + 40
\end{align*}
\]

\[-\frac{1}{18} q + 120 = \frac{1}{30} q + 40\]

\[ q = 900 \quad \text{equilibrium quantity} \]

\[ p = 70 \quad \text{equilibrium price} \]
3. If a tax of $1.5 per unit is to be imposed on the manufacturer, how will the equilibrium price be affected if the demand remains the same?

The supply after tax is

\[ P = \frac{1}{30} q + 40 + 1.5 = \frac{1}{30} q + 41.5 \]

\[ P = \frac{1}{18} q + 120 \]

\[ P = \frac{1}{30} q + 41.5 \]

\( q \approx 88.3 \) equilibrium quantity

\( P \approx 70.94 \) equilibrium price.

Cost, Revenue and Profit

The revenue resulting from business activities is the total payment received from the sale of goods and services to customers. The profit is the difference between the revenue and the cost of making a product or providing a service.

\[ \text{Profit} = \text{Revenue} - \text{Cost} \]

If the profit is negative, we refer to a loss. To break-even means to make neither profit nor a loss. Thus break-even occurs when the revenue equals the cost, or the profit is zero.

**Example 12:** You see a good opportunity to open a business as a private garbage collector during the Toronto strike. However the license costs $500 and you estimate that the cost per bag of garbage is $2.4 for transportation and disposal. Now you charge $4.9 per bag of garbage. Let \( x \) be the number of bags of garbage you collected.

1. What is the revenue \( R(x) \)?

\[ g_{TR} = 4.9 x \]

2. What is your total cost \( C(x) \)?

\[ g_{TC} = 500 + 2.4 x \]

3. What is your profit \( P(x) \)?

\[ \text{Profit} = g_{TR} - g_{TC} = 2.5 x - 500 \]
4. How many bags do you have to collect to break-even?
   
   To break-even, \( TR = TC \) or Profit = 0
   
   \[ 2.5x - 500 = 0 \]
   
   \[ x = 200 \]
   
   200 bags has to be collected to break-even.

**Quadratic Functions and Nonlinear Systems**

To prepare for this topic, please read sections §3.3, §3.5 and §3.6 in the textbook.

*In real-life applications, the relationship between two quantities is often best modeled by a curve rather than a line. The simplest function with a graph that is not a line is a quadratic function.*

**Definition**

A function is a **quadratic function** if and only if \( f(x) \) can be written in the form \( f(x) = ax^2 + bx + c \), where \( a, b \) and \( c \) are constants and \( a \neq 0 \).

**Features of Quadratic Functions**

1. The graph of a quadratic function \( y = ax^2 + bx + c \) is a parabola. If \( a > 0 \) the parabola opens upward (concave up) and if \( a < 0 \) the parabola opens downward (concave down).

2. The vertex is the turning point of the parabola. Its \( x \)-coordinate is \(-\frac{b}{2a}\). If \( a > 0 \) the vertex is the lowest point on the parabola. This means that the parabola has a minimum value at this point. If \( a < 0 \), the vertex is the highest point on the parabola, so the function has a maximum value at the vertex.

3. The parabola is symmetry with respect to the vertical line through the vertex, that is \( x = -\frac{b}{2a} \).

4. The \( x \)-intercept points are those intersection points (if there are any) of the parabola with the \( x \) axis. These \( x \)-intercepts can be found by setting \( y = 0 \), i.e., \( ax^2 + bx + c = 0 \) and solving the equation by either factoring or using the quadratic formula. The \( x \)-intercepts are
   
   \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
   
   If the discriminant \( b^2 - 4ac \) is positive, there are two \( x \)-intercepts. If it is zero, there is one single \( x \)-intercept. If it is negative, there are no \( x \)-intercepts (so the parabola does not meet the \( x \)-axis).

5. The \( y \)-intercept occurs when \( x = 0 \), so \( y = c \).
Example 13: The demand function for a publisher's line of cookbook is \( p = 50 - \frac{q}{1000} \), where \( p \) is the price in dollars per book when \( q \) books are demanded by consumers. Find the level of production that will maximize the publisher's total revenue.

1. What is the revenue \( R(q) \) if \( q \) books are demanded at a price \( p \)?

\[
\frac{d}{dq} R = p \frac{d}{dq} q = (50 - \frac{q}{1000}) \frac{d}{dq} q = 50 q - \frac{q^2}{1000}
\]

\[
\frac{d}{dq} = - \frac{q^2}{1000} + 50 q
\]

2. Plot the graph of the revenue \( R(q) \).
   - It is a quadratic function
   - \( a = -\frac{1}{1000} < 0 \) opens down
   - \( y \)-intercepts
     - Let \( y = 0 \) we have
       \[
       - \frac{q^2}{1000} + 50 q = 0
       \]
       \[
       q(-\frac{q}{1000} + 50) = 0
       \]
       \[
       q = 0 \text{ or } q = 50,000
       \]
   - The vertex
     \[
     q = -\frac{b}{2a} = 25,000.
     \]

3. Find \( q \) to maximize \( R \).
   
   Since the curve opens downward, the vertex will give us the maximum.
   
   So the maximum revenue happens when \( q = 25,000 \).
   
   and the max \( R = 625,000 \).
4. Suppose there is a fixed cost of $195,000 to set up the book and a printing cost of $6 per book. Find the break-even price and break-even quantity.

Now \[ \frac{\partial TC}{\partial q} = 195,000 + 6q \]

To find the break-even point, we solve

\[
\begin{align*}
\frac{\partial TR}{\partial q} &= \frac{-1}{1000} q^2 + 50q \\
\frac{\partial TC}{\partial q} &= 195,000 + 6q
\end{align*}
\]

break-even if \[ \frac{\partial TR}{\partial q} = \frac{\partial TC}{\partial q} \]

\[ \frac{-1}{1000} q^2 + 50q = 195,000 + 6q \]

i.e. \[ \frac{-1}{1000} q^2 + 44q - 195,000 = 0 \]

Solving by quadratic formula, we have \[ q_1 = 5000, \quad q_2 = 36,000 \]

\[ Example 14: \text{(Example 1 in Section 3.5)} \]

Solve system of equations.

\[
\begin{align*}
x^2 - 2x + y - 7 &= 0 \\
3x - y + 1 &= 0
\end{align*}
\]

Solving \( \text{\#} \) for \( y \)

\[ y = 3x + 1 \]

Substituting into \( \text{\#} \)

\[ x^2 - 2x + (3x + 1) - 7 = 0 \]

i.e. \[ x^2 + x - 6 = 0 \]

\[ (x+3)(x-2) = 0 \]

\[ x_1 = -3, \quad x_2 = 2 \]

if \( x_1 = -3 \), \[ y_1 = 3(-3) + 1 = -8 \]

if \( x_2 = 2 \), \[ y_2 = 3(2) + 1 = 7 \]

there are two solutions \( x_1 = -3, \quad y_1 = -8 \) and \( x_2 = 2, \quad y_2 = 7 \).