Week 4: Limits and Continuity

Goals:

- Introduce limits
- Study limit properties. Learn to evaluate limits
- Introduce continuity

Suggested Textbook Readings: Chapter 10: §10.1 - §10.3.

Practice Problems:

- §10.1: 3, 11, 23, 29, 31, 35, 39, 43
- §10.2: 2, 7, 13, 19, 23, 29, 35, 45
- §10.3: 9, 10
- §10.5 Review: 7, 15, 19, 25, 29, 41

Answers for §10.2 Problem 2

(a) \( \lim_{x \to 0^-} f(x) = 0 \)
(b) \( \lim_{x \to 0^+} f(x) = -\infty \)
(c) \( \lim_{x \to 0} f(x) \) doesn’t exist
(d) \( \lim_{x \to -\infty} f(x) = +\infty \)
(e) \( \lim_{x \to 1} f(x) = 2 \)
(f) \( \lim_{x \to -\infty} f(x) = 1 \)
(g) \( \lim_{x \to 2^+} f(x) = 1 \)
Limits

To prepare for this topic, you should read section §10.1 and §10.2 in the textbook.

We say that the limit of a function \( f(x) \), as \( x \) approaches a point \( a \), is equal to a number \( L \) if the \( f(x) \) can be made as close to \( L \) as desired by making \( x \) sufficiently close to \( a \), but not equal to \( a \). We write this as

\[
\lim_{x \to a} f(x) = L
\]

If there is no such number, we say that the limit does not exist.

For example, if we look at the function \( f(x) = 2 + x \) as \( x \) approaches 3, we get the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>2.99</th>
<th>2.999</th>
<th>2.9999</th>
<th>3</th>
<th>3.0001</th>
<th>3.001</th>
<th>3.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = 2 + x )</td>
<td>4.99</td>
<td>4.999</td>
<td>4.9999</td>
<td>5</td>
<td>5.0001</td>
<td>5.001</td>
<td>5.01</td>
</tr>
</tbody>
</table>

From the table, the closer \( x \) gets to 3 from either side, the closer \( f(x) \) gets to 5. This suggests that

\[
\lim_{x \to 3} f(x) = 5
\]

Example 1: Estimate the limit \( \lim_{x \to 0} \frac{3x}{x} \)

\[ \text{If } x \neq 0 \quad \frac{3x}{x} = 3 \quad \text{the graph of } y = \frac{3x}{x} \text{ is} \]

\[ \lim_{x \to 0} \frac{3x}{x} = 3 \]

Example 2: Use a table to estimate the following limit

\[
\lim_{x \to 0} \frac{e^x - 1}{x} = 1
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>-0.0001</th>
<th>0</th>
<th>0.0001</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^x - 1 )</td>
<td>0.951626</td>
<td>0.995017</td>
<td>0.9995</td>
<td>0.99995</td>
<td>1.00005</td>
<td>1.0005</td>
<td>1.005017</td>
<td>1.0517</td>
<td></td>
</tr>
</tbody>
</table>
**Estimate Limit from Graphs**

**Example 3:** Let \( f(x) = \frac{x^3 - 1}{x - 1} \). Draw the graph of \( y = f(x) \). Use the graph to estimate
\[
\lim_{x \to 1} \frac{x^3 - 1}{x - 1}.
\]

Since \( x^3 - 1 = (x-1)(x^2 + x + 1) \), we have
\[
\frac{x^3 - 1}{x - 1} = \frac{(x-1)(x^2 + x + 1)}{x - 1} = x^2 + x + 1 \quad \text{if } x \neq 1.
\]

So
\[
\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3
\]

**Example 4:** Estimate \( \lim_{x \to 2} f(x) \) and \( \lim_{x \to 1} f(x) \), where the graph of \( f \) is given below.

\[ \lim_{x \to 2} f(x) = f(2) = 0.5 \]

\[ \lim_{x \to 1} f(x) = 2 \]
LIMITS DO NOT ALWAYS EXIST

Example 5: Consider the following limits:

1. \[ \lim_{x \to 0} \frac{1}{x^2} \]

   The graph of \( y = \frac{1}{x^2} \) is 
   when \( x \) approaches 0, the value of \( \frac{1}{x^2} \) increases without bound 
   so the limit doesn't exist.

2. \[ \lim_{x \to 0} \frac{|x|}{x} \]

   \[ \frac{|x|}{x} = \begin{cases} 
   1 & x > 0 \\
   -1 & x < 0 
   \end{cases} \]

   It is not defined at \( x = 0 \).

   The graph of \( y = \frac{|x|}{x} \) is 
   if \( x \) approaches 0 from the left, the value of \( \frac{|x|}{x} \) is -1, and if 
   \( x \) approaches 0 from the right, the value of \( \frac{|x|}{x} \) is 1.

   So the limit doesn't exist.

3. \[ \lim_{x \to 0} \frac{1}{x} \]

   The graph of \( y = \frac{1}{x} \) is 
   if \( x \) approaches 0 from the left, the value of \( \frac{1}{x} \) approaches \(-\infty\), 
   and if \( x \) approaches 0 from the right, the value of \( \frac{1}{x} \) approaches \( +\infty \).

   So the limit doesn't exist.
One-Sided Limits

In examining the limits we just calculated, we see it can sometimes be informative to determine the one-sided limit of a function; for these limits we consider values of the function as \( x \) approaches from only one direction. The left-side limit (or the right-side limit) (if exist) as \( x \) approaches \( a \) from left (or right) is denoted by \( \lim_{x \to a^-} f(x) \) (or \( \lim_{x \to a^+} f(x) \)).

The limit of \( f(x) \) exists and \( \lim_{x \to a} f(x) = L \) if and only if

- both left and right-side limits exist, and
- \( \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \)

**Example 6:** For the function \( f \) given in the figure below, find the following limits.

![Graph of a function with points labeled a, b, c, d, e, and asymptotes]

(a) \( \lim_{x \to a^-} f(x) = \infty \)  
(b) \( \lim_{x \to a^+} f(x) = \infty \)

(c) \( \lim_{x \to b} f(x) = f(b) \)  
(d) \( \lim_{x \to c^-} f(x) = 8 \)

(e) \( \lim_{x \to c^+} f(x) = 3 \)  
(f) \( \lim_{x \to d^-} f(x) = -2 \)

(g) \( \lim_{x \to d^+} f(x) = -2 \)  
(h) \( \lim_{x \to e} f(x) = 5 \)

**Vertical Asymptote**

The line \( x = a \) is a vertical asymptote for the graph of the function \( f \) if and only if

\[ \lim_{x \to a^+} f(x) = \pm \infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm \infty \]

[Textbook, page 590 (12th), page 600 (13th)]
Finding a limit

<table>
<thead>
<tr>
<th>Properties of Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. If ( f(x) = c ) is a constant function, ( \lim_{x \to a} f(x) = c )</td>
</tr>
<tr>
<td>2. ( \lim_{x \to a} x^n = a^n ), for any positive integer ( n )</td>
</tr>
<tr>
<td>3. ( \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) )</td>
</tr>
<tr>
<td>4. ( \lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) )</td>
</tr>
<tr>
<td>5. ( \lim_{x \to a} c f(x) = c \lim_{x \to a} f(x) )</td>
</tr>
<tr>
<td>6. If ( f(x) ) is a polynomial function, then ( \lim_{x \to a} f(x) = f(a) )</td>
</tr>
<tr>
<td>7. ( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} ), if ( \lim_{x \to a} g(x) \neq 0 )</td>
</tr>
<tr>
<td>8. ( \lim_{x \to a} \sqrt{f(x)} = \sqrt{\lim_{x \to a} f(x)} )</td>
</tr>
</tbody>
</table>

Example 7: (a) \( \lim_{x \to 2} (x^2 + x) = \lim_{x \to 2} x^2 + \lim_{x \to 2} x = 4 + 2 = 6 \)

(b) \( \lim_{x \to 2} [(x + 1)(x - 3)] = \lim_{x \to 2} (x + 1) \lim_{x \to 2} (x - 3) = (2 + 1)(2 - 3) = -3 \)

(c) \( \lim_{x \to 1} \frac{2x^2 + x - 3}{x^3 + 4} = \frac{\lim_{x \to 1} (2x^2 + x - 3)}{\lim_{x \to 1} (x^3 + 4)} = \frac{2 + 1 - 3}{1 + 4} = 0 \)

(d) \( \lim_{x \to -2} \sqrt{x^2 - 1} = \sqrt{\lim_{x \to -2} (x^2 - 1)} = \sqrt{3} \)
Example 8: (a) Find \( \lim_{x \to -1} \frac{x^2 - 1}{x + 1} \).

Note that \( x^2 - 1 = (x-1)(x+1) \).

\[
\frac{x^2 - 1}{x + 1} = \frac{(x-1)(x+1)}{x+1} = x-1 \quad \text{if} \quad x \neq -1
\]

Then \( \lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} (x-1) = -2 \).

(b) Find \( \lim_{x \to 4} \frac{x - 4}{\sqrt{x+5} - 3} \).

\[
\lim_{x \to 4} \frac{x - 4}{\sqrt{x+5} - 3} = \lim_{x \to 4} \frac{x - 4}{\sqrt{x+5} - 3} \cdot \frac{\sqrt{x+5} + 3}{\sqrt{x+5} + 3} = \lim_{x \to 4} \frac{(x-4)(\sqrt{x+5} + 3)}{\sqrt{x+5}^2 - 3^2} = \lim_{x \to 4} \frac{(x-4)(\sqrt{x+5} + 3)}{x-4} = \lim_{x \to 4} (\sqrt{x+5} + 3) = 6
\]

If a limit cannot be evaluated by direct substitution, then we need to manipulate the expression of the function to find the limit if it exists. A fundamental result is the following:

If \( f \) and \( g \) are two functions for which \( f(x) = g(x) \), for all \( x \neq a \), then

\[
\lim_{x \to a} f(x) = \lim_{x \to a} g(x)
\]
Example 9: If \( f(x) = x^2 + x + 1 \), find

\[
\lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h}
\]

\[
f(x+h) = (x+h)^2 + (x+h) + 1
\]

\[
f(x) = x^2 + x + 1
\]

\[
f(x+h) - f(x) = (x+h)^2 + (x+h) + 1 - (x^2 + x + 1)
\]

\[
= 2xh + h^2 + h = h(2x + h + 1)
\]

Then

\[
\frac{f(x+h) - f(x)}{h} = \frac{h(2x + h + 1)}{h} = 2x + h + 1 \quad \text{if} \quad h \neq 0
\]

\[
\lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h} = \lim_{{h \to 0}} (2x + h + 1) = 2x + 1
\]
Limits at Infinity

We can also consider the behaviour of the function when \( x \) is very large. We say that the limit of \( f(x) \) as \( x \) approaches infinity is equal to \( L \), denoted by

\[
\lim_{x \to \infty} f(x) = L
\]

if \( f(x) \) becomes arbitrarily close to \( L \) when \( x \) is arbitrarily large. We make the similar definition for \( x \to -\infty \) when \( x \) is negative. For example, in compound interest we have seen that the value of \( \left( 1 + \frac{1}{n} \right)^n \) approaches \( e \) when \( n \) approaches \( +\infty \). It can be denoted by

\[
\lim_{n \to +\infty} \left( 1 + \frac{1}{n} \right)^n = e
\]

Example 10: \( \lim_{x \to \infty} \frac{1}{x} = 0 \)

In fact \( \lim_{x \to \infty} \frac{1}{x^p} = 0 \) for \( p > 0 \). This can be used to find limits of rational functions.

Example 11: \( \lim_{x \to \infty} \frac{4x^2 + 5}{2x^2 + 1} \)

By the graph, \( \lim_{x \to \infty} \frac{4x^2 + 5}{2x^2 + 1} = 2 \).

\[
\lim_{x \to \infty} \frac{4x^2 + 5}{2x^2 + 1} = \lim_{x \to \infty} \frac{(4x^2 + 5)/x^2}{(2x^2 + 1)/x^2} = \lim_{x \to \infty} \frac{4 + \frac{5}{x^2}}{2 + \frac{1}{x^2}} = \frac{4}{2} = 2
\]

HORIZONTAL ASYMPTOTE

Let \( f \) be a function. The line \( y = b \) is a horizontal asymptote for the graph of \( f \) if and only if at least one of the following is true:

\[
\lim_{x \to +\infty} f(x) = b, \text{ or } \lim_{x \to -\infty} f(x) = b
\]

[Textbook, page 591 (12th), page 602 (13th)]
Example 12: (a) \[ \lim_{x \to -\infty} \frac{x}{(3x - 1)^2} = \lim_{x \to -\infty} \frac{x}{9x^2 - 6x + 1} = \lim_{x \to -\infty} \frac{\frac{x}{x^2}}{\frac{9x^2 - 6x + 1}{x^2}} = \frac{1}{9} \]

(b) \[ \lim_{x \to -\infty} \frac{x^5 - x^4}{x^4 - x^3 + 2} = \lim_{x \to -\infty} \frac{(x^5 - x^4)}{(x^4 - x^3 + 2)} \cdot \frac{1}{x^4} = \lim_{x \to -\infty} \frac{x - 1}{1 - \frac{1}{x} + \frac{2}{x^2}} = \infty \]

(c) \[ \lim_{x \to \infty} (\sqrt{x^2 + x} - x) = \lim_{x \to \infty} \left( \sqrt{x^2 + x} - x \right) \cdot \frac{\sqrt{x^2 + x} + x}{\sqrt{x^2 + x} + x} = \lim_{x \to \infty} \frac{2x}{\sqrt{x^2 + x} + x} = \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + x} + x} = \frac{1}{2} \]
Continuity

To prepare for this topic, you should read section §10.3 in the textbook.

A function is continuous at $a$ if and only if the following three conditions are met:

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

If $f$ is not continuous at $a$, then $f$ is said to be discontinuous at $a$, and $a$ is called a point of discontinuity of $f$.

Concept Question 1 At what points does the function fail to be continuous?

![Graph showing points of discontinuity](image)

- at $x = a$, the function is not defined
- at $x = c$, $\lim_{x \to c^-} f \neq \lim_{x \to c^+} f$
- at $x = e$, $\lim_{x \to e^-} f = \lim_{x \to e^+} f \neq f(e)$

The polynomial, exponential and logarithmic functions are continuous on their domains.
Example 13: Find all points of discontinuity.

(a) \( f(x) = \frac{x^2 - 3}{x^2 + 2x - 8} \)

\( x^2 - 3 \) and \( x^2 + 2x - 8 \) are continuous functions.

\( f(x) \) is not defined if \( x^2 + 2x - 8 = 0 \), that is, if \( x = -4 \) or \( x = 2 \).

So \( f \) is not continuous at \( x = -4 \) and \( x = 2 \).

(b) \( f(x) = \begin{cases} 
  x + 2, & \text{if } x > 2, \\
  3, & \text{if } x = 2, \\
  x^2, & \text{if } x < 2.
\end{cases} \)

\( y = x + 2 \) and \( y = x^2 \) are continuous. The only possible place that \( f(x) \) is discontinuous is at \( x = 2 \).

Since

\( \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4 \)

\( \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x + 2 = 4 \)

but \( f(2) = 3 \).

We have \( \lim_{x \to 2} f(x) \neq f(2) \)

So the function is not continuous at \( x = 2 \).