Week 6: Differentiation Rules and Applications

Goals:

- Study Product Rule and Quotient Rule.
- Study applications of derivatives to Economics: marginal cost and marginal revenue


Practice Problems
- §11.2: 17, 27, 35, 39, 47, 53, 55, 57, 63, 69, 75, 79, 83, 85, 89
- §11.3: 9, 11, 17, 21, 25*, 27, 45
- §11.4: 13, 15, 23, 27, 37, 41, 50, 53, 71

Remarks:
(1) §11.3 - 25 in the 12th edition and 13th edition are given below.

- (12th) \( r = 250q + 45q^2 - q^3; q = 5, q = 10, q = 25 \)
- (13th) \( r = 240q + 40q^2 - 2q^3; q = 10, q = 15, q = 20 \)

(2) The answer to §11.5 - 50 is \( f'(-1) = 0.5 \).
Differentiation Rules

To prepare for this topic, please read section §11.2 in the textbook.

Example 1: Prove the Constant Factor Rule.
If \( f(x) \) is differentiable, then \( cf(x) \) is differentiable and

\[
\frac{d}{dx} [cf(x)] = c \frac{d}{dx} f(x)
\]

Let \( g(x) = cf(x) \), then \( g(x+h) = cf(x+h) \)

\[
\frac{g(x+h) - g(x)}{h} = \frac{cf(x+h) - cf(x)}{h} = c \cdot \frac{f(x+h) - f(x)}{h}
\]

\[
\frac{d}{dx} g(x) = \frac{d}{dx} [cf(x)] = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} c \cdot \frac{f(x+h) - f(x)}{h} = c \cdot \frac{d}{dx} f(x)
\]

Constant Factor Rule

If \( f(x) \) is differentiable, then \( cf(x) \) is differentiable and

\[
\frac{d}{dx} [cf(x)] = c \frac{d}{dx} f(x)
\]

Sum or Difference Rule

If \( f \) and \( g \) are differentiable, then \( f \pm g \) are differentiable and

\[
\frac{d}{dx} [f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)
\]
Example 2: Differentiate the following functions

(1) \( f(x) = \frac{3x^2 - 2}{x} \)

\[ f'(x) = 3x - 2x^{-1} \]
\[ f' = 3 - 2(-1)x^{-2} = 3 + 2x^{-2} \]  

[Example 7 in Section 11.2]

(2) \( f(x) = (x + 1)(x^2 - 5x + 7) \)

\[ f'(x) = x^2 - 5x + 7x + x^2 - 5x + 7 = x^3 - 4x^2 + 2x + 7 \]
\[ f' = 3x^2 - 8x + 2 \]
The Product Rule and The Quotient Rule

To prepare for this topic, please read section §11.4 in the textbook.

Example 3: Let \( g(x) = x^2 + 3x \), \( h(x) = 4x + 5 \) and \( f(x) = (x^2 + 3x)(4x + 5) \). Find \( g'(x) \), \( h'(x) \) and \( f'(x) \).

\[
\begin{align*}
f &= (x^2 + 3x)(4x + 5) = 4x^3 + 5x^2 + 12x^2 + 15x = 4x^3 + 17x^2 + 15x \\
f' &= 12x^2 + 34x + 15 \\
g(x) &= x^2 + 3x \\
g' &= 2x + 3 \\
h(x) &= 4x + 5 \\
h' &= 4 \\
g'h + gh' &= (2x+3)(4x+5) + (x^2+3x)4 \\
&= 8x^2 + 10x + 12x + 15 + 4x^2 + 12x \\
&= 12x^2 + 34x + 15 .
\end{align*}
\]

If \( f = gh \), then

\[
(fg)' = g'h + gh'
\]

The Product Rule

If \( f \) and \( g \) are differentiable functions, then the product \( fg \) is differentiable, and

\[
\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)
\]

[Textbook, Page 507 (12th), Page 518 (13th)]
Example 4: (Example 2 in Section 11.4) If \( y = (x^{2/3} + 3)(x^{-1/3} + 5x) \), find \( dy/dx \).

\[
\frac{dy}{dx} = \left( x^{2/3} + 3 \right)' \left( x^{-1/3} + 5x \right) + \left( x^{2/3} + 3 \right) \left( x^{-1/3} + 5x \right)'
\]

\[
= \frac{2}{3} x^{-1/3} \left( x^{-1/3} + 5x \right) + \left( x^{2/3} + 3 \right) \left( -\frac{1}{3} x^{-4/3} + 5 \right)
\]

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**The Quotient Rule**

If \( f \) and \( g \) are differentiable functions, then the quotient \( \frac{f}{g} \) is differentiable, and

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'g - fg'}{g^2}
\]

[Textbook, Page 510 (12th), Page 521 (13th)]

Example 5: (Example 5 in Section 11.4) If \( f(x) = \frac{4x^2 + 3}{2x - 1} \), find \( f''(x) \).

\[
f'' = \frac{\left( 4x^2 + 3 \right)'(2x - 1) - (4x^2 + 3)(2x - 1)''}{(2x - 1)^2}
\]

\[
= \frac{8x(2x - 1) - (4x^2 + 3)2}{(2x - 1)^2}
\]
The Derivative as a Rate of Change

To prepare for this topic, please read section §11.3 in the textbook.

If \( y = f(x) \), then the average rate of change of \( y \) with respect to \( x \) over the interval from \( x \) to \( x + \Delta x \) is

\[
\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

and the instantaneous rate of change of \( y \) with respect to \( x \) is

\[
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{dy}{dx}
\]

Therefore the instantaneous rate of change is also the slope of the tangent line to the graph of \( y = f(x) \) at that point. For convenience, we usually refer to the instantaneous rate of change simply as the rate of change. The interpretation of a derivative as a rate of change is extremely important.

**Example 6:** Interpret the following statements using derivatives.

(a) An object is moving along a line at a speed of 2 meters per second with position function \( s = f(t) \), where \( s \) is in meters and \( t \) is in seconds.

\[
\frac{ds}{dt} = f'(t) = 2 \text{ m/s}
\]

(b) A spherical balloon is being filled with air. The rate of change of the volume with respect to its radius is 16 ft\(^3\) per ft of change in radius.

let \( V \) be the volume and \( r \) be the radius. Then \( V = V(r) \)

\[
\frac{dV}{dr} = 16 \text{ ft}^3/\text{ft}
\]

(c) Babies are born at a rate of one every 20 minutes and people are dying at a rate of one every 30 minutes in a city. Assume \( P = P(t) \) is the population at time \( t \), where \( t \) is in years.

birth rate : \( \gamma_b = 3/\text{hr} \)

death rate : \( \gamma_d = 2/\text{hr} \)

Growth rate : \( \gamma_b - \gamma_d = 1/\text{hr} = 38760/\text{year} \)

since 1 year = 8760 hours

Then the rate of change of \( P \) with respect to \( t \) is

\[
\frac{dP}{dt} = 38760 \text{ /year}
\]
Example 7: (Example 4 in Section 11.3) Let \( p = 100 - q^2 \) be the demand function for a manufacturer’s product. Find the rate of change of price \( p \) per unit with respect to quantity \( q \). How fast is the price changing with respect to \( q \) when \( q = 5 \)? Assume that \( p \) is in dollars.

- Want to find the rate of change of the price \( p \) w.r.t \( q \), i.e. \( \frac{dp}{dq} \).
- Since \( p = 100 - q^2 \), \( \frac{dp}{dq} = -2q \).
- If \( q = 5 \), \( \frac{dp}{dq} = -10 \) $/unit.$

Example 8: Suppose that a position function of an object moving along a straight line is \( s = f(t) = 2t^2 \), where \( t \) is in seconds and \( s \) is in meters.

(a) Find the velocity of the object at \( t = 1 \).

\[
\frac{ds}{dt} = 4t \quad \text{if} \quad t=1 \quad \frac{ds}{dt} = 4 \quad \text{m/s}
\]

(b) How far does the object travel in the next 1 second, i.e., what is \( \Delta s = f(2) - f(1) \)?

What is the average velocity \( \frac{\Delta s}{\Delta t} \) over the time interval \([1, 2]\)?

\[
f(2) = 8 \quad f(1) = 2
\]

\[
\Delta s = f(2) - f(1) = 6 \quad \text{m}
\]

\[
\frac{\Delta s}{\Delta t} = \frac{6}{1} = 6 \quad \text{m/s}
\]

(c) Fill the following table.

<table>
<thead>
<tr>
<th>( \Delta t )</th>
<th>( \Delta s = f(1 + \Delta t) - f(1) )</th>
<th>( \frac{ds}{dt} \Delta t )</th>
<th>( \frac{\Delta s}{\Delta t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>0.5</td>
<td>( f(1.05) - f(1) = 2.5 )</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>0.1</td>
<td>( f(1.01) - f(1) = 0.42 )</td>
<td>0.4</td>
<td>4.2</td>
</tr>
<tr>
<td>0.01</td>
<td>( f(1.001) - f(1) = 0.0402 )</td>
<td>0.04</td>
<td>4.02</td>
</tr>
<tr>
<td>0.001</td>
<td>( f(1.0001) - f(1) = 0.004002 )</td>
<td>0.004</td>
<td>4.002</td>
</tr>
</tbody>
</table>

(d) What conclusion can you draw from above table?

If \( \Delta t \) is small, \( \frac{\Delta s}{\Delta t} \approx \frac{ds}{dt} \) and \( \Delta s \approx \frac{ds}{dt} \Delta t \).
We can obtain a relatively quick and simple way to approximating the change in $y$ due to a small change $\Delta x$. If $\Delta x$ is very small, then

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$$

So,

$$\Delta y \approx \frac{dy}{dx} \Delta x$$

[Textbook, Page 499 (12th), Page 511 (13th)]

This can be illustrated by the following graph

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**Example 9: (Example 2 in Section 11.3)** Suppose that $y = f(x)$ and $\frac{dy}{dx} = 8$ when $x = 3$. Estimate the change in $y$ if $x$ changes from 3 to 3.5.

$$\Delta x = 3.5 - 3 = 0.5$$

$$\Delta y = \frac{dy}{dx} \Delta x = 8 \times 0.5 = 4$$
The derivative of a function $f(x)$ at $x = a$ represents the instantaneous rate of change of the function with respect to $x$ at $x = a$. If $c = f(q)$ is a cost function which gives the total cost $c$ if producing and marketing $q$ units of a product, the rate of change of the cost $c$ with respect to $q$ is called the **marginal cost**.

$$\text{marginal cost} = \frac{dc}{dq}$$

We interpret marginal cost as the approximate cost of one additional unit of output.

**Example 10:** The cost (in thousands of dollars) of manufacture a product is given by

$$c = 150 + 2250x - 0.02x^2$$

(a) Find the average cost function $\bar{c}$.

$$\bar{c} = \frac{c}{x} = \frac{150}{x} + 2250 - 0.02x$$

(b) Find the marginal cost function and use it to estimate how fast the cost is increasing when $x = 4$. Compare this with the exact cost of producing the 5th product.

$$\frac{dc}{dx} = 2250 - 0.04x$$

If $x = 4$, the marginal cost is

$$\frac{dc}{dx} = 2249.84$$

It gives an approximation of the cost for producing the 5th unit.

The actual cost

$$c(5) - c(4) \approx 2249.82$$
Example 11: (Example 7 in Section 11.3) If a manufacturer’s average-cost equation is
\[ \bar{c} = 0.0001q^2 - 0.02q + 5 + \frac{5000}{q} \]
find the marginal-cost function. What is the marginal cost when 50 units are produced?

The cost function
\[ C = \bar{c} q = 0.0001 \frac{q^3}{q} - 0.02 q^2 + 5 q + 5000 \]

\[ \frac{dc}{dq} = 0.0003 \frac{q^2}{q} - 0.04 q + 5 \]

If \( q = 50 \),
\[ \frac{dc}{dq} = 3.75 \]

We can extend the ideal of marginal cost to include other functions, like revenue and profit.

A revenue or profit function specifies the total revenue \( R(q) \) or profit \( P(q) \) as a function of the number of items \( q \). The derivatives, \( R'(q) \) and \( P'(q) \) of these functions are called the marginal revenue and marginal profit functions.

\[ \text{marginal revenue} = \frac{dR}{dq}, \quad \text{marginal profit} = \frac{dP}{dq} \]

Marginal revenue (profit) indicates the rate at which revenue (profit) changes with respect to the number of units \( q \) sold. We interpret it as the approximate revenue (profit) received from selling one additional unit of output.
Example 12: (Example 8 in Section 11.3) If the demand equation for a manufacturer's product is

\[ p = \frac{1000}{q + 5} \]

where \( p \) is in dollars.

(a) Find the revenue function.

\[ R = pq = \frac{1000q}{q + 5} \]

(b) Find the marginal-revenue function and evaluated it when \( q = 45 \).

\[ \frac{dR}{dq} = \frac{1000(\frac{q}{q + 5}) - 1000 \frac{q + 1}{(q + 5)^2}}{\frac{(q + 5) - (q + 1)}{(q + 5)^2}} = \frac{5000}{(q + 5)^2} \]

\[ \text{if } q = 45, \quad \frac{dR}{dq} = 2 \]

Example 13: You operate an product customizing service (a typical customized product might have a custom color case and a personalized logo). The cost in dollars to refurbish \( x \) products in a month is given by

\[ c(x) = 0.25x^2 + 40x + 500, \quad 0 < x \leq 80 \]

You charge customer $70 per product for the work.

(a) Find the revenue function and the profit function.

\[ \text{if } x \text{ units are refurbished} \]

\[ \text{Revenue: } R = 70x \]

\[ \text{Profit: } P = R - C = -0.25x^2 + 30x - 500 \]

(b) Find the marginal revenue and marginal profit.

\[ \frac{dR}{dx} = 70 \quad \frac{dP}{dx} = -0.5x + 30 = -\frac{1}{2}x + 30 \]

\[ \text{Note: } \quad \frac{dP}{dx} = \frac{dR}{dx} - \frac{dc}{dx} \]

\[ \text{Marginal profit} = \text{marginal revenue} - \text{marginal cost}. \]
(c) Compute the marginal revenue and marginal profit if you have refurbished 20 units.

\[
\frac{dR}{dx} = 70 \quad \frac{dP}{dx} = 20 \quad \frac{dc}{dx} = 50.
\]

For refurbishing the 21st unit, we expect to have revenue $70. It will cost $50, so our profit is approximately $20.

(d) Draw the graphs of the revenue and cost function in the same coordinate plane. Interpret the graphs.

\[
R = 70x \\
c = 0.25x^2 + 40x + 500
\]

*Break even happens at* $x = 20$ and $x = 100$.

At $x = 20$:

\[
\frac{dR}{dx} = 70 \\
\frac{dc}{dx} = 50.
\]

\[
\frac{dP}{dx} = 20, \text{ so our profit increases}
\]

At $x = 30$:

\[
\frac{dR}{dx} = 70 \\
\frac{dc}{dx} = 55
\]

\[
\frac{dP}{dx} = 15, \text{ our profit increases at a slower rate than before}
\]

At $x = 60$:

\[
\frac{dR}{dx} = 60 \quad \frac{dc}{dx} = 60. \text{ So } \frac{dP}{dx} = 0. \text{ Selling one more unit will bring in a profit. On the graph, the since } \frac{dc}{dx} \text{ is the slope of the tangent to the cost curve. At } x = 60, \frac{dR}{dx} = \frac{dc}{dx} \text{ means the tangent line at } x = 60 \text{ of the cost curve is parallel to the revenue line.}
\]

If $x > 60$, $\frac{dc}{dx} > \frac{dR}{dx}$. So $\frac{dP}{dx} < 0$. For example:

At $x = 70$:

\[
\frac{dc}{dx} = 75 \quad \text{but} \quad \frac{dR}{dx} = 70. \text{ So } \frac{dP}{dx} = -5. \text{ That is, we'll lose } 5 \text{ if we take } 1 \text{ unit from the sale of } 76 \text{th unit. Therefore if } x > 60, \text{ our profit will decrease.}