Week 7: Chain Rule and Elasticity of Demand

Goals:

- Study Chain Rule
- Study applications of derivatives to Economics: Elasticity of Demand

Suggested Textbook Readings: Chapter 11: §11.5, Chapter 12: §12.3

Practice Problems

- §11.5: 5, 7, 17, 21, 27, 31, 35, 41, 51, 55, 59, 67, 71
- Chapter 11 Review: 19, 31, 33, 49, 51, 59
- §12.3: 5, 11, 15, 17
The Chain Rule

To prepare for this topic, please read section §11.5 in the textbook.

The Chain Rule gives the derivative of a composition of two functions in terms of the derivatives of the component functions.

Example 1: If \( y = (x^3 + x^2 + 1)^3 \), find \( y' \).

Let \( u = x^3 + x^2 + 1 \) \( \frac{du}{dx} = 3x^2 + 2x \)

\( y = u^3 \) \( \frac{dy}{du} = 3u^2 \)

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

\[ = 3u^2 \cdot (3x^2 + 2x) \]

\[ = 3(x^3 + x^2 + 1)^2 (3x^2 + 2x) \]

---

The Chain Rule

If \( y \) is a differentiable function of \( u \) and \( u \) is a differentiable function of \( x \), then \( y \) is a differentiable function of \( x \) and

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

[Textbook, Section 11.5]

MATH 126
Example 2: (Example 1 in Section 11.5) (a) If \( y = 2u^2 - 3u - 2 \) and \( u = x^2 + 4 \), find \( \frac{dy}{dx} \).

\[
\frac{dy}{du} = 4u - 3 \quad \frac{dy}{dx} = 2x
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (4u - 3)(2x) = \left[4(x^2 + 4) - 3\right]2x
\]

\[
= 2x \left(4x^2 + 13\right)
\]

(b) If \( y = \sqrt{w} \) and \( w = 7 - t^3 \), find \( \frac{dy}{dt} \).

\[
\frac{dy}{dw} = \frac{1}{2} w^{-\frac{1}{2}} = \frac{1}{2\sqrt{w}} \quad \frac{dw}{dt} = -3t^2
\]

\[
\frac{dy}{dt} = \frac{dy}{dw} \cdot \frac{dw}{dt} = \frac{1}{2\sqrt{w}} \cdot (-3t^2)
\]

\[
= -\frac{3t^2}{\sqrt{7 - t^3}}
\]
The Generalized Power Rule

If \( u \) is a differentiable function of \( x \) and \( n \) is any real number, then

\[
\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}
\]

[Textbook, Section 11.5]

Example 3: Compute the following derivatives.

(a) \( \frac{d}{dx} \left( \frac{1}{\sqrt{x^2 - 6}} \right) \)

Let \( y = \frac{1}{\sqrt{x^2 - 6}} = (x^2 - 6)^{-\frac{1}{2}} \)

\[
\frac{dy}{dx} = -\frac{1}{2} (x^2 - 6)^{-\frac{3}{2}} \cdot \frac{d}{dx} (x^2 - 6)
\]

\[
= -\frac{1}{2} (x^2 - 6)^{-\frac{3}{2}} (2x)
\]

(b) \( \frac{d}{dx} [(2x^2 + x)^3] \)

\[
= 3(2x^2 + x)^2 \cdot (2x^2 + x)
\]

\[
= 3(2x^2 + x)^2 (4x + 1)
\]

Example 4: If \( y = (x + 10)^3 \sqrt{1 - x^2} \), find \( y' \).

\[
y' = 3(x + 10)^2 \cdot \sqrt{1 - x^2} + (x + 10)^3 \cdot \frac{1}{2} (1 - x^2)^{-\frac{1}{2}} \cdot (-2x)
\]
The rate of change of revenue with respect to the number of employees is called the **marginal-revenue product**. It approximates the change in revenue that results when a manufacturer hires an extra employee.

**Example 5**: (Example 8 in Section 11.5) A manufacturer determines that $m$ employees will produce a total of $q$ units of a product per day, where

$$q = \frac{10m^2}{\sqrt{m^2 + 19}}$$

If the demand equation for the product is $p = 900/(q+9)$, determine the marginal-revenue product when $m = 9$.

We want to find $\frac{dR}{dm}$ at $m = 9$.

The revenue $R = pq = \frac{900}{q+9} \cdot q = \frac{900q}{q+9}$ and $q = \frac{10m^2}{\sqrt{m^2+19}}$

so $R$ is a function of $m$ through $q$. Then

$$\frac{dR}{dm} = \frac{dR}{dq} \cdot \frac{dq}{dm}$$

$$\frac{dR}{dq} = \frac{900(q+9) - 900q}{(q+9)^2} = \frac{8100}{(q+9)^2}$$

$$\frac{dq}{dm} = \frac{20m\sqrt{m^2+19} - 10m^2 \cdot \frac{1}{2} (m^2+19)^{-\frac{1}{2}} \cdot 2m}{m^2+19} = \frac{20m\sqrt{m^2+19} - \frac{10m^3}{\sqrt{m^2+19}}}{m^2+19}$$

at $m = 9$. $q = 81$

$$\frac{dR}{dq} = 1.$$

and $\frac{dq}{dm} = 10.71$

so $\frac{dR}{dm} = 10.71$. 


Example 6: Suppose a company’s monthly revenue $R$ is given by a function of the unit price $p$ and the price $p$ is a function of the monthly sales $q$. Suppose if $q = 500$ units,
\[
\frac{dR}{dp} = 35, \quad \text{and} \quad \frac{dp}{dq} = -15
\]

(a) Interpret the two rates $\frac{dR}{dp}$ and $\frac{dp}{dq}$.

- If 500 units are sold, $\frac{dR}{dp} = 35$ means the revenue will increase $\$ 35$ if the price is increased by $\$ 1$.
- $\frac{dp}{dq}$ represents the rate of change of price with respect to the units sold.
  - The price will drop $\$ 15$ if one more unit is sold.

(b) Find the marginal revenue $\frac{dR}{dq}$.

\[
\frac{dR}{dq} = \frac{dR}{dp} \cdot \frac{dp}{dq} = (35)(-15) = -525
\]

Look again at the way the term "$dp$" appeared to cancel in the differential formula $\frac{dR}{dq} = \frac{dR}{dp} \cdot \frac{dp}{dq}$. In fact the chain rule tells us more about the derivative notation: Suppose $y$ is a function of $x$. Then we may think $x$ as a function of $y$. One has
\[
\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}, \quad \text{if} \quad \frac{dy}{dx} \neq 0
\]

Notice again that $\frac{dy}{dx}$ behaves like a fraction.
Elasticity of Demand

To prepare for this topic, please read section §12.3 in the textbook.

In economics, elasticity of demand is defined as the ratio of the percent change in quantity demanded and the percent change in the price. It used to measure the responsiveness of the demand to the price change.

\[
\text{elasticity of demand} = \frac{\text{percentage change in quantity}}{\text{percentage change in price}}
\]

We call the demand is inelastic if the changes in price result only modest changes in the quantity demanded, the demand is elastic if a slight change in price leads to a sharp change in the quantity demanded.

let the demand function be \( p = f(q) \). If \( q \) units are demanded, the unit price is \( p = f(q) \) and if \( q + h \) units are demanded, the unit price is \( f(q + h) \).

\[
\text{percentage change in quantity} = \frac{(q+h) - q}{q} \times 100\% = \frac{h}{q} \times 100\%
\]

\[
\text{percentage change in price} = \frac{f(q+h) - f(q)}{f(q)} \times 100\%
\]

\[
\text{elasticity of demand} = \frac{\frac{h}{q} \times 100\%}{\frac{f(q+h)-f(q)}{f(q)} \times 100\%} = \frac{f(q)}{f(q+h)-f(q)} \times \frac{dP}{dQ}
\]

if \( h \to 0 \). Then \( \frac{f(q+h)-f(q)}{h} \to \frac{df}{dq} \).

So elasticity of demand \( \frac{dP}{dQ} = \frac{f(q)}{f(q+h)-f(q)} \times \frac{df}{dq} \).

If \( p = f(q) \) is a differentiable demand function, the point elasticity of demand, denoted by \( \eta \), at \( (q, p) \) is given by

\[
\eta = \eta(q) = \frac{p}{dq} = \frac{p}{q} \cdot \frac{dq}{dp}
\]

We say that the demand is elastic if \(|\eta| > 1\), is inelastic if \(|\eta| < 1\), and has unit elasticity if \(|\eta| = 1\).

Notice that if \( q = g(p) \), then

\[
\eta = \eta(p) = \frac{p}{q} \cdot \frac{dq}{dp} = \frac{p}{g(p)} \cdot g'(p) = p \cdot \frac{g'(p)}{g(p)}
\]
Example 7: (Example 2 in Section 12.3) The demand function for a product is

\[ q = p^2 - 40p + 400, \quad \text{where} \quad q > 0 \quad (0 < p < 20) \]

(a) Find the point elasticity of demand \( \eta(p) \).

\[ \frac{d \frac{\partial q}{\partial p}}{dp} = 2p - 40 \]

\[ \eta (p) = \frac{p}{q} \cdot \frac{d \frac{\partial q}{\partial p}}{dp} = \frac{p}{p^2 - 40p + 400} \cdot (2p - 40) = \frac{p(2p - 40)}{p^2 - 40p + 400} \]

(b) Determine the point elasticity of demand when \( p = 15 \)

\[ \eta (15) = -6 \]

(c) If the price is lowered by 2% (from $15 to $14.70), use the answer to (b) to estimate the corresponding percentage change in quantity sold.

\[ \text{percentage change in price} = -2\% \]

\[ \text{percentage change in quantity} = \eta(15) \cdot \text{percentage change in price} \]

\[ = -6 \cdot (-2\%) = +12\% \]

(d) Will the change in part (c) result in an increase or decrease in revenue?

The demand is elastic, so lowering the price increases the revenue.

Check: at price 15, quantity demanded \( q = 25 \), revenue \( R = 375 \)
at price 14.70, quantity demanded \( q = 28.09 \times 28 \), revenue \( R = 411.6 \).
So revenue increases.

(e) Find the range of price for which the demand is elastic and the range for which the demand is inelastic.

(whence \( 0 < p < 20 \))

We first find the point of unit elasticity, that is, find \( p \) such that \( \eta(p) = -1 \)

Solving \( \frac{p(2p - 40)}{p^2 - 40p + 400} = -1 \), we have \( p = \frac{20}{3} \), \( p = 20 \) (omitted)

- If \( p < \frac{20}{3} \), \( \eta(p) > -1 \) or \( |\eta(p)| < 1 \), so the demand is inelastic.
- If \( p > \frac{20}{3} \), \( \eta(p) < -1 \) or \( |\eta(p)| > 1 \), so the demand is elastic.
Example 8: (Textbook, page 542 (12th), page 553 (13th)) Let the demand equation be \( p = mq + b \) where \( p \) is the unit price and \( q \) is the quantity demanded when the price is \( p \).

(a) Find the point elasticity of demand \( \eta(p) \).

\[
\eta(p) = \frac{\frac{\partial p}{\partial q}}{\frac{\partial q}{\partial p}} = \frac{p}{m} = \frac{\frac{p}{m}}{\frac{1}{m} (\frac{p-b}{m})} = \frac{p}{p-b}
\]

(b) Is \( \eta(p) \) increasing or decreasing?

\[
\eta'(p) = \frac{-b}{(p-b)^2} < 0
\]

so \( \eta(p) \) decreases.

(c) Find the range of price for which the demand is elastic and the range for which the demand is inelastic.

let \( \eta(p) = -1 \).

\[
\frac{p}{p-b} = -1 \quad p = \frac{1}{2} b = \frac{b}{2}
\]

if \( p = \frac{b}{2} \), the demand has unit elasticity.

if \( p < \frac{b}{2} \), \( \eta(p) > -1 \), \( |\eta(p)| < 1 \) the demand is inelastic.

if \( p > \frac{b}{2} \), \( \eta(p) < -1 \), \( |\eta(p)| > 1 \) the demand is elastic.