Week 8: Additional Differentiation Topics

Goals:

- Study the derivatives of exponential and logarithmic Functions.
- Study the implicit differentiation and higher order derivatives.


Practice Problems

- §12.1: 9, 13, 27, 29, 33, 37, 45, 49, 51 (different in two editions)
- §12.2: 9, 11, 13, 21, 23, 29, 33, 41, 43
- §12.4: 17, 19 (13th edition), 21, 25, 29, 33
- §12.7: 5, 9, 21, 29, 33, 39
- Review Problems: 7, 9, 15, 17, 21, 25, 31, 33 (12th edition), 37, 45, 49

Remarks:


\[ \ln(xy) = e^{xy} \]

- Review Problems - 33 (12th edition): Evaluate $y'$ at the given value of $x$

\[ y = e^{x+\ln(1/x)}, \quad x = e \]
Derivatives of Exponential Functions

At this point, we know how to differentiate functions of $x$ involving powers and radicals. We now turn to the derivatives of the logarithmic and exponential functions.

**Example 1:** Find the slope of the tangent of the graph of $y = e^x$ at $x = 0$. Recall the limit (Week 4, Example 11)

$$
\lim_{x \to 0} \frac{e^x - 1}{x} = 1
$$

$$
f'(0) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^h - 1}{h} = 1
$$

so the slope of the tangent at $x = 0$ is 1.

**Example 2:** Use the definition to find the derivative of the function $y = e^x$.

$$
y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}
$$

$$
= \lim_{h \to 0} e^x \left( e^h - 1 \right) = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x
$$
The Derivative of the Function $y = e^x$

$$\frac{d}{dx}(e^x) = e^x$$

**Example 3:** Differentiate the following functions

(a) $y = \frac{x}{e^x}$

$$y' = \frac{1 \cdot e^x - x \cdot e^x}{(e^x)^2} = \frac{e^x (1-x)}{(e^x)^2} = \frac{1-x}{e^x}$$

(b) $y = e^x + e^x$

$$y' = 0 + e^x = e^x$$

(c) $y = e^{x^3 + 3x}$

$$y' = e^{x^3 + 3x} \cdot (3x^2 + 3) = 3(x^2 + 1)e^{x^3 + 3x}$$

The Derivative of the Function $y = a^x$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

**Example 4:** (Example 5 in Section 12.2) Find $\frac{d}{dx}(e^x + x^e + 2\sqrt{x})$.

$$\frac{d}{dx}(e^x + x^e + 2\sqrt{x}) = e^x e^{-1} + 2 \cdot \frac{1}{2} x^{-\frac{1}{2}}$$

$$= e^x e^{-1} + \frac{2 \cdot \sqrt{x} \cdot \ln 2}{2 \cdot \sqrt{x}}$$
Derivatives of Logarithmic Functions

To prepare for this topic, please read section §12.1 in the textbook.

The Derivative of the Natural Logarithm

\[ \frac{d}{dx} (\ln |x|) = \frac{1}{x}, \text{ for } x \neq 0 \]

If \( y = f(x) \), then
\[ \frac{dy}{dx} = \frac{1}{f'(x)}. \]  

Let \( y = \ln x \). Then \( x = e^y \)
\[ \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x} \]

**Example 5:** Differentiate the following functions
(a) \( y = \frac{\ln x}{x^2} \).
\[ y' = \frac{\frac{1}{x} \cdot x^2 - (\ln x) \cdot 2x}{(x^2)^2} = \frac{1 - 2 \ln x}{x^3} \]

(b) \( y = \ln(x^2 + 1) \).
\[ y' = \frac{1}{x^2 + 1} \cdot 2x = \frac{2x}{x^2 + 1} \]

(c) \( y = 2(\ln x)^3 + 5x \)
\[ y' = 2 \cdot 3 (\ln x)^2 \cdot \frac{1}{x} + 5 \]

The Derivatives of Logarithmic functions to the Base \( b \)

\[ \frac{d}{dx} (\log_b x) = \frac{1}{x \ln b} \]
Implicit Differentiation

To prepare for this topic, please read section §12.4 in the textbook.

So far we have described functions by expressing one variable explicitly in terms of another variable; for example, \( y = x^2 \), or \( y = \frac{x + 1}{\sqrt{x} - 1} \), or, in general \( y = f(x) \). Here \( y \) is said to be an explicit function of \( x \). But other functions are defined implicitly by a relation between \( x \) and \( y \) such as \( x^2 + y^2 = 25 \), or \( x^3 + y^3 = xy^2 \). Such an equation is said to give \( y \) as an implicit function of \( x \). To differentiate an implicit function we use implicit differentiation: differentiating both sides of the equation with respect to \( x \) and then solving the resulting equation for \( y' \).

**Example 6:** Find the derivative \( y' \) at \( x = \sqrt{2} \) if \( y \) is defined by the half-circle \( x^2 + y^2 = 4 \), \( y \geq 0 \)

(a) Write \( y \) explicitly in terms of \( x \) and differentiate the resulting function.

\[
\begin{align*}
y &= \sqrt{4 - x^2} \\
y' &= \frac{1}{2} (4 - x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{4 - x^2}}
\end{align*}
\]

\( \delta x = \sqrt{2} \)

\[
y' &= \frac{-\sqrt{2}}{\sqrt{4} - 2} = -1
\]

(b) Differentiate the function given by \( x^2 + y^2 = 4 \) implicitly.

- Differentiating both sides with respect to \( x \), \( y \) is treated as a function of \( x \), we have

\[
2x + 2y \cdot y' = 0
\]

- Solving \( y' \) from above equation, we obtain

\[
y' = -\frac{x}{y}
\]

Then at \( x = \sqrt{2} \), \( y = \sqrt{2} \) so

\[
y' = -\frac{\sqrt{2}}{\sqrt{2}} = -1
\]
Example 7: (Example 2 in Section 12.4) Find \( \frac{dy}{dx} \) if \( x^3 + 4x^2y - 27 = y^4 \).

- diff. both sides w.r.t \( x \), treating \( y \) as a function of \( x \).

\[
3x^2 + 4(2xy + x^2 \cdot y') - 0 = 4y^3 \cdot y'
\]

- solving \( y' \)

\[
y' = -\frac{3x^2 + 8xy}{4x^2 - 4y^3}
\]

Example 8: (Problems 12.4 - 30, page 548 (12th), page 560 (13th)) Find an equation of the tangent line to the curve \( y^2 + xy - x^2 = 5 \) at the point \((4,3)\).

- find \( y' \) at \( x = 4 \), \( y = 3 \)

\[
2y \cdot y' + y + x \cdot y' - 2x = 0
\]

\[
y' = \frac{2x - y}{2y + x}
\]

\( y' = \frac{1}{2} \) \( x = 4 \), \( y = 3 \), \( y' = \frac{1}{2} \)

- the equation is

\[
y = \frac{1}{2} (x - 4) + 3 = \frac{1}{2} x - 1
\]
Higher-Order Derivatives

To prepare for this topic, please read section §12.7 in the textbook.

If \( f \) is a differentiable function, then its derivative \( f' \) is also a function, so \( f' \) may have a derivative of its own, denoted by \( (f')' = f'' \). This new function \( f'' \) is called the second derivative of \( f \) because it is the derivative of the derivative of \( f \). The second derivative of \( y = f(x) \) is denoted by \( f''(x) \) or \( \frac{d^2 y}{dx^2} \). Similarly the derivative of the second derivative is called the third derivative, denoted by \( f'''(x) \). In general the \( n \)-th derivative of \( y = f(x) \) is denoted by \( y^{(n)} \) or

\[
f^{(n)}(x), \quad \frac{d^n y}{dx^n}, \quad \frac{d^n}{dx^n} f(x)
\]

Example 9: Find the second derivative of \( f(x) = 5x^2 + \sqrt{x} \).

\[
y' = 10x + \frac{1}{2} x^{-\frac{1}{2}}
\]
\[
y'' = 10 + \frac{1}{2} \cdot (-\frac{1}{2}) x^{-\frac{3}{2}} = 10 - \frac{1}{4} x^{-\frac{3}{2}}
\]

Example 10: (Problems 12.7 - 38, page 560 (12th), page 571 (13th)) If \( p = 400 - 40q - q^2 \) is a demand equation, how fast is marginal revenue changing when \( q = 4 \).

We want to find \( \frac{d}{dq} MR \), i.e. \( R'' \) at \( q = 4 \).

The revenue function is

\[
R = pq = (400 - 40q - q^2)q = 400q - 40q^2 - q^3
\]
\[
R' = 400 - 80q - 3q^2
\]
\[
R'' = -80 - 6q
\]

at \( q = 4 \)

\[
R'' = -104
\]
Recall that the first derivative of $f$ tells us where the graph of $f$ is rising (where $f' > 0$) and where it is falling (where $f' < 0$). The second derivative tells in what direction the graph of $f$ curves or bends.

The following graphs are concave upward:

![Concave Up Graphs](image)

The following graphs are concave downward:

![Concave Down Graphs](image)

The graph of a function $f$ is called **concave up** on an interval $I$ if it lies above all of its tangents on $I$. It is called **concave down** on $I$ if it lies below all of these tangents. A point $P$ on a curve is called a **point of inflection** if the curve changes its concavity.

**Criteria for Concavity**

- If $f''(x) > 0$ for all $x$ in an interval $I$, then the graph of $f$ is concave up on $I$.
- If $f''(x) < 0$ for all $x$ in an interval $I$, then the graph of $f$ is concave down on $I$.

[Textbook, page 581 (12th), page 592 (13th)]