Week 9: Derivatives and the graphs of functions

Goals:

- Study L'Hôpital's Rule
- Study the relation between the values of \( f' \) and \( f'' \) and the graphs of \( f \).
- Introduce relative and absolute Extrema.


Practice Problems

- §13.3: 3, 13, 19, 27, 31, 33, 39, 49, 57, 63, 65, 70
- §13.4: 5, 11, 13
L'Hôpital's Rule

In Chapter 10 we studied how to take limits even when those limits had an inconvenient form such as the ratio of two functions that were both going to zero. Solving such limits required algebraic manipulation. In this section we learn a more powerful and often easier way to resolve this kind of limit, L'Hôpital’s Rule. This rule only works when the limit is in the form of a fraction where the numerator and denominator are both going to zero or are both going to infinity.

**L'Hôpital’s Rule**

If \( f \) and \( g \) are differentiable, \( f(a) = g(a) = 0 \), then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

providing the limit on the right exists.

**Example 1:** Calculate \( \lim_{x \to 0} \frac{e^x - 1}{x} \)

\[
\text{If } x = 0, \quad e^x - 1 = 0 \quad \text{and} \quad x = 0. \quad \text{The limit is } \frac{0}{0}.
\]

\[
\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1
\]

**Example 2:** Calculate \( \lim_{x \to 1} \frac{x^3 + x^2 - 5x + 3}{x^4 - x^3 - x + 1} \)

\[
\text{At } x = 0, \quad x^3 + x^2 - 5x + 3 = 0 \quad \text{and} \quad x^4 - x^3 - x + 1 = 0. \quad \text{So it is } \frac{0}{0}.
\]

\[
\lim_{x \to 1} \frac{x^3 + x^2 - 5x + 3}{x^4 - x^3 - x + 1} = \lim_{x \to 1} \frac{3x^2 + 2x - 5}{4x^3 - 3x^2 - 1}
\]

\[
= \lim_{x \to 1} \frac{6x + 2}{12x^2 - 6x} = \frac{8}{6} = \frac{4}{3}
\]

(by direct substitution)
L'Hôpital rule works in other cases besides 0/0 forms. It works on expressions of the form ∞/∞ and also on the form 0/0 and ∞/∞ when $x \to \infty$.

**Example 3:** (a) $\lim_{x \to 0} x \ln x$

\[
\lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0} -x = 0
\]

(b) $\lim_{x \to \infty} \frac{x^2}{e^x}$

\[
\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0
\]

(c) *Show that* $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

Let $y = \left(1 + \frac{1}{x}\right)^x$. Then $\ln y = \ln \left(1 + \frac{1}{x}\right)^x = x \ln \left(1 + \frac{1}{x}\right)$

\[
\lim_{x \to \infty} \ln y = \lim_{x \to \infty} x \ln \left(1 + \frac{1}{x}\right) = \lim_{x \to \infty} \ln \frac{1 + \frac{1}{x}}{x}
\]

L'Hôpital's rule

\[
= \lim_{x \to \infty} \frac{\frac{-1}{x^2}}{-\frac{1}{x^2}} = \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} = \lim_{x \to \infty} \frac{x}{x + 1}
\]

L'Hôpital's rule

\[
= \lim_{x \to \infty} \frac{1}{1 + 0} = 1
\]

So $\ln y = 1$. Then $\lim_{x \to \infty} \ln y = \ln \left(\lim_{x \to \infty} y\right) = 1$

\[
\lim_{x \to \infty} y = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e
\]

For practice, try some questions in Chapter 10.
Relative Extrema and Curve Sketching

To prepare for this topic, please read sections §13.1, §13.3, and §13.4 in the textbook. The information about the graph of a function $f$ provided by the signs of $f'$ and $f''$ on an interval $(a, b)$ is expressed in the following table.

<table>
<thead>
<tr>
<th>$f'(x) &gt; 0$ on $(a, b)$</th>
<th>$f$ increasing on $(a, b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x) &lt; 0$ on $(a, b)$</td>
<td>$f$ decreasing on $(a, b)$</td>
</tr>
<tr>
<td>$f''(x) &gt; 0$ on $(a, b)$</td>
<td>$f$ concave up on $(a, b)$</td>
</tr>
<tr>
<td>$f''(x) &lt; 0$ on $(a, b)$</td>
<td>$f$ concave down on $(a, b)$</td>
</tr>
</tbody>
</table>

Concept Question 1 The graph of the function $y = f(x)$ is drawn below:

![Graph of function](image)

True or False:

1. $f(x) > 0$ on $(x_1, b)$  
   F since $f(x) < 0$ over the interval $(x_1, A)$
2. $f'(x) < 0$ on $(b, 0)$  
   T
3. $f'(x) > 0$ on $(0, x_2)$  
   F since $f' < 0$ over the interval $(0, c)$
4. $f'(x) > 0$ on $(c, d)$  
   T
5. $f''(x) > 0$ on $(c, d)$  
   F since the curve concave down ($f'' < 0$) over the interval $(c, d)$
6. $f''(x) > 0$ on $(0, x_2)$  
   T
7. $f''(b) > 0$  
   F in fact $f''(b) < 0$
8. $f'(0) < 0$  
   T
9. $f'(c) = 0$  
   T
Concept Question 2  True or False:

1. If \( f(x) > 0 \) then \( f'(x) > 0 \).  \( F \).  \( \text{Example: } y = x^2 > 0 \) but \( y' = 2x \) and at \( x = -1 \) \( y' < 0 \).

2. If \( f'(x) > 0 \) then \( f(x) \) is increasing.  \( T \)

3. If \( f(x) \) is increasing then \( f'(x) \) is increasing.  \( F \).  \( \text{Example: } y = \ln x \) is increasing but \( y' = \frac{1}{x} \) is decreasing.

4. If \( f'(x) = 0 \) then \( f''(x) = 0 \).  \( F \)

5. If \( f'(x) \) is increasing then \( f''(x) > 0 \).  \( T \)

Example 4: (Problems 13.1 - 68) If \( c = 3q - 3q^2 + q^3 \) is a cost function, when is the marginal cost increasing?

The marginal cost
\[
MC = c' = 3 - 6q + 3q^2
\]
To check the increasing/decreasing property of \( MC \), we take the derivative of \( MC \)
\[
MC' = c'' = -6 + 6q
\]
Let \( MC' = 0 \). Then \( q = 1 \)
when \( q < 1 \), \( MC' < 0 \). So \( MC \) decreases.
when \( q > 1 \), \( MC' > 0 \). So \( MC \) increases.

Example 5: (Problems 13.3 - 67) Show that the graph of the demand equation \( p = \frac{100}{q + 2} \) is decreasing and concave up for \( q > 0 \).

\[
p' = \frac{-100}{(q+2)^2} < 0 \quad \text{, } \quad \text{P decreases}
\]
\[
p'' = \frac{200}{(q+2)^3} > 0 \text{ for } q > 0 \quad \text{, } \quad \text{P is always concave up.}
\]
All the indicators above deal with none-zero values of $f'(x)$ and $f''(x)$. What is distinctive about the zero values of these derivatives?

**Critical Points**

For $x = a$ in the domain of $f$, if either

- $f'(a) = 0$, or
- $f'(a)$ does not exist.

then $x = a$ is called a **critical value** for $f$. If $x = a$ is a critical value, then the point $(a, f(a))$ is called a critical point for $f$. [Textbook, page 570 (12th), page 580 (13th)]

Note that $f(a)$ must be defined for $a$ to be a critical point. For example, $x = 0$ is not a critical point of $f(x) = 1/x$. We do not consider endpoints of the interval where $f(x)$ is defined to be critical points. For example, $x = 0$ is not a critical point of $y = \sqrt{x}$, since the domain of $f(x)$ is $[0, \infty)$.

**Example 6:** The graph of the function $y = f(x)$ is drawn below. Mark the critical points on the graph by circling them.
A function has a **relative maximum** at \( x = a \) if there is an open interval containing \( a \) on which \( f(a) \geq f(x) \) for all \( x \) in the interval. The relative maximum value is \( f(a) \). A function has a **relative minimum** at \( x = a \) if there is an open interval containing \( a \) on which \( f(a) \leq f(x) \) for all \( x \) in the interval. The relative minimum value is \( f(a) \).

A function has a **absolute maximum** at \( x = a \) if \( f(a) \geq f(x) \) for all \( x \) in the domain of \( f \). A function has a **absolute minimum** at \( x = a \) if \( f(a) \leq f(x) \) for all \( x \) in the domain of \( f \).

[Textbook, page 569 (12th), page 579 (13th)]

**Example 7:** *On the graph below, mark the points at which the relative maxima and minima occur, as well as the location of absolute maximum and minimum on the interval \([a, b]\).*

How are the set of critical points and the set of relative extrema related?

```
relative extrema are critical points, but a critical point may not be a relative max/min.
```
A Necessary Condition for Relative Extrema
If \( f \) has a relative extremum at \( x = a \), then \( f'(a) = 0 \) or \( f'(a) \) does not exist.
If \( f(x) \) has a relative max/min at \( a \), then \( a \) must be a critical point, but not all critical points give rise to a relative maximum or minimum.

First Derivative Test
Suppose \( f \) is continuous on an open interval \( I \) that contains the critical value \( x = a \) and \( f \) is differentiable on \( I \), except possibly at \( a \).

1. If \( f'(x) \) changes from positive to negative as \( x \) increases through \( a \), then \( f \) has a relative maximum at \( a \).
2. If \( f'(x) \) changes from negative to positive as \( x \) increases through \( a \), then \( f \) has a relative minimum at \( a \).
3. If \( f'(x) \) does not change sign as \( x \) increases through \( a \), then \( f \) has no maximum or minimum at \( a \).

[Textbook, page 570 (12th), page 580 (13th)]

Example 8: Complete the table

<table>
<thead>
<tr>
<th></th>
<th>sign of ( f' ) to the left of ( a )</th>
<th>sign of ( f' ) to the right of ( a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>relative minimum at ( a )</td>
<td>( - )</td>
<td>( + )</td>
</tr>
<tr>
<td>relative maximum at ( a )</td>
<td>( + )</td>
<td>( - )</td>
</tr>
</tbody>
</table>
Example 9: Find all critical points of the function \( f(x) = x^4 - 4x^3 - 8x^2 - 1 \). Use the first derivative test to show whether each critical point is a relative maximum or a relative minimum.

\[
f'(x) = 4x^3 - 12x^2 - 16x = 4x(x^2 - 3x - 4) = 4x(x-4)(x+1) = 0
\]

\( x = 0, \ x = 4, \ x = -1 \) are critical values.

<table>
<thead>
<tr>
<th></th>
<th>(-10)</th>
<th>(-1)</th>
<th>0</th>
<th>4</th>
<th>(+\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4x )</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( x - 4 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( x + 1 )</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

We may also use the second derivative test to determine if a critical point is a relative maximum or minimum.

**Second Derivative Test**

Suppose \( f''(a) = 0 \).

1. If \( f''(a) > 0 \) then \( f \) has a relative minimum at \( a \).
2. If \( f''(a) < 0 \) then \( f \) has a relative maximum at \( a \).
3. If \( f'(a) = 0 \) and \( f''(a) = 0 \) then the test is inconclusive.

[Textbook, page 588 (12th), page 598 (13th)]
The second derivative test does not apply when \( f''(a) = 0 \). For example, let \( f(x) = x^3 \), \( g(x) = x^4 \), and \( h(x) = -x^4 \). It is easy to calculate that \( f'(0) = g'(0) = h'(0) = 0 \). So \( x = 0 \) is a critical value of \( f \), \( g \), and \( h \). Furthermore, \( f''(0) = g''(0) = h''(0) = 0 \). But at \( x = 0 \), \( f \) has no relative max/min, \( g \) has a relative minimum at \( x = 0 \), and \( h \) has a relative maximum. In such case, the first derivative test should be used to analyze what is happening at \( a \).

**Example 10:** Test the following for relative maxima and minima. Use the second derivative test, if possible.

(a) [Example 3 in Section 13.1] \( y = x^2 e^x \)

\[
\begin{align*}
y' &= 2xe^x + x^2 e^x = xe^x(2 + x) = 0, \quad x = 0, x = -2 \\
y'' &= 2e^x + 2xe^x + 2xe^x + x^2 e^x = e^x(2 + 4x + x^2) \\
\text{at } x = 0 \quad &y'' = 2 > 0, \quad \text{so the function has a relative min at } x = 0 \\
\text{at } x = -2 \quad &y'' = -2e^2 < 0, \quad \text{it is a relative max.}
\end{align*}
\]

(b) [Example 2 in Section 13.3] \( y = 6x^4 - 8x^3 + 1 \)

\[
\begin{align*}
y' &= 24x^3 - 24x^2 = 24x^2(x - 1) = 0 \\
&x = 0, \quad x = 1 \quad \text{are critical values.} \\
y'' &= 72x^2 - 48x = 24x(3x - 2) \\
\text{at } x = 1. \quad &y'' = 24 > 0 \\
\quad &\text{it is a relative min.} \\
\text{at } x = 0. \quad &y'' = 0. \quad \text{The 2nd derivative test is inconclusive.} \\
\quad &\text{Now use 1st derivative test}
\end{align*}
\]

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 24x^2 )</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( x - 1 )</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( y' )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y'' )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( x = 0 \) is neither a max nor a min.

MATH 126
Curve Sketching

With the help of the first and second derivatives, one can give a rough sketch of the graph of a function.

Example 11: (Example 4 in Section 13.1) Let \( y = 2x^2 - x^4 \). Use derivative information to sketch the graph of the function.

- **Intercepts**
  - **x-intercepts**: let \( y = 0 \). \( 2x^2 - x^4 = 0 \). \( x^2(2-x^2) = 0 \).
    - \( x = 0 \). \( x = \sqrt{2} \). \( x = -\sqrt{2} \).
  - **y-intercepts**: let \( x = 0 \). \( y = 0 \).

- **Increasing /Decreasing, Max/Min**
  
  \[
  y' = 4x - 4x^3 = 4x(1-x^2) = 0
  \]
  
  \( x = 0 \). \( x = 1 \). \( x = -1 \) are critical values

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\infty)</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>(+\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4x )</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( (1-x^2) )</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( y' )</td>
<td>+</td>
<td>-</td>
<td>relative max</td>
<td>relative min</td>
<td>relative max</td>
</tr>
</tbody>
</table>

  **relative max** points are \((-1,1), (1,1)\)
  **relative min** point is \((0,0)\).

- **Concavity**

  \[
  y'' = 4 - 12x^2 = 4(1-3x^2) = 0
  \]
  
  \( x = \frac{1}{\sqrt{3}} \). \( x = -\frac{1}{\sqrt{3}} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\infty)</th>
<th>(-\frac{1}{\sqrt{3}})</th>
<th>( \frac{1}{\sqrt{3}} )</th>
<th>(+\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y'' )</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( y )</td>
<td>( \cup ) inflection point ( \cup ) inflection point ( \cup )</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

  **inflection points** are \((-\frac{1}{\sqrt{3}}, \frac{5}{9}), (\frac{1}{\sqrt{3}}, \frac{5}{9})\)

- **Sketch the Graph**

  ![Graph of the function](image)
Example 12: (Example 2 in Section 13.3) Sketch the curve $y = 6x^4 - 8x^3 + 1$

- **Intercepts**
  - $y$-intercept: let $x = 0$, $y = 1$.
  - It is not easy to find $x$-intercepts.

- **Increasing/decreasing, max/min**
  - $y' = 24x^3 - 24x^2 = 24x^2(x-1)$
  - Critical values are $x = 0$, $x = 1$.

<table>
<thead>
<tr>
<th></th>
<th>$-\infty$</th>
<th>$0$</th>
<th>$1$</th>
<th>$+\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x-1$</td>
<td>$-$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
<tr>
<td>$24x^2$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
<tr>
<td>$y'$</td>
<td>$-$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
<tr>
<td>$y$</td>
<td>$-$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Relative min point $(1, -1)$

- **Concavity**
  - $y'' = 72x^2 - 48x = 24x(3x-2)$
  - Critical values are $x = 0$, $x = \frac{2}{3}$

<table>
<thead>
<tr>
<th></th>
<th>$-\infty$</th>
<th>$0$</th>
<th>$\frac{2}{3}$</th>
<th>$+\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x-2$</td>
<td>$-$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
<tr>
<td>$24x$</td>
<td>$-$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
<tr>
<td>$y''$</td>
<td>$+$</td>
<td>$-$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>$y'$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Inflection points are $(0, 1), (\frac{2}{3}, -\frac{5}{27})$

- **Sketch**

![Graph of the function](image)