Instructions: This is a three-hour exam. There are 4 problems worth a total of 50 marks as indicated in the box below. Problem 1 consists of 10 questions each worth 2 marks. The remaining problems consist of five questions, (a-e) each worth 2 marks. Answer all questions in the space provided. If you need more room, answer on the back of the previous page. Where appropriate, you must provide clear explanations.

When you are asked to find a numerical answer by looking at a graph, a reasonable estimate is all that is required, but it is important that the geometric construction behind the answer be clearly illustrated on your graph and explained in the space provided. When the answer to the question is an IRRATIONAL NUMBER, please provide the exact answer and the numerical approximation.

You are allowed to use gold sticker calculators and rulers.

Please Note: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.
Problem 1. Multiple Choice (5 questions, 2 marks each)

Each question has four possible answers, labelled (A), (B), (C), and (D). Choose the most appropriate answer. Write your answer in the space provided. Write clearly using UPPERCASE letters. Illegible answers will be marked incorrect. You do not need to justify your answer.

(1) When \( x = 1 \) there are two points on the curve of equation \( y^2 x - 2 x^2 + \frac{1}{x} - 3 = 0 \). The slopes at these two points are:

(A) \( \frac{1}{2}, -\frac{1}{4} \)

(B) \( \frac{1}{4}, \frac{1}{4} \)

(C) \( \frac{1}{4}, -\frac{1}{4} \)

(D) \( \frac{1}{2}, -\frac{1}{2} \)

ANSWER TO (1): C

SOLUTION: First, we find the two points. When \( x = 1 \), the equation becomes

\[
y^2 - 2 + 1 - 3 = 0
\]
\[
\Rightarrow y^2 = 4
\]
\[
\Rightarrow y = \pm 2
\]
so the two points are \((1, -2)\) and \((1, 2)\). Now, let’s take the derivative of both sides:

\[
\frac{d}{dx} \left( y^2 x - 2 x^2 + \frac{1}{x} - 3 \right) = \frac{d}{dx} 0
\]
\[
2y \frac{dy}{dx} x + y^2 - 4x - \frac{1}{x^2} = 0
\]
Plugging in our first point, \( x = 1, y = -2 \), gives

\[
2(-2) \frac{dy}{dx} (1) + (-2)^2 - 4(1) - \frac{1}{(1)^2} = 0,
\]
\[
\Rightarrow -4 \frac{dy}{dx} + 4 - 4 - 1 = 0
\]
\[
\Rightarrow \frac{dy}{dx} = \frac{-1}{4}
\]
Plugging in our second point, \( x = 1, y = 2 \), gives

\[
2(2) \frac{dy}{dx} (1) + (2)^2 - 4(1) - \frac{1}{(1)^2} = 0,
\]
\[
\Rightarrow 4 \frac{dy}{dx} + 4 - 4 - 1 = 0
\]
\[
\Rightarrow \frac{dy}{dx} = \frac{1}{4}
\]
(2) Find the solutions of the equation $2^{x^2-5x} \cdot \frac{1}{64} = 0$. Both solutions belong to which interval?

(A) \[ \begin{array}{cccccc}
-5 & -3 & -2 & -1 & 0 & 4 & 5 & 6 \\
\end{array} \]

(B) \[ \begin{array}{cccccc}
-5 & -3 & -2 & -1 & 0 & 4 & 5 & 6 \\
\end{array} \]

(C) \[ \begin{array}{cccccc}
-5 & -3 & -2 & -1 & 0 & 4 & 5 & 6 \\
\end{array} \]

(D) \[ \begin{array}{cccccc}
-5 & -3 & -2 & -1 & 0 & 4 & 5 & 6 \\
\end{array} \]

ANSWER TO (2): ___ C ___

SOLUTION: First, note that $\frac{1}{64} = 2^{-6}$, and so we want to find the values of $x$ such that

$$2^{x^2-5x} \cdot 2^{-6} = 0.$$ 

This amounts to finding values of $x$ such that $x^2 - 5x = -6$. Let’s add 6 to both sides, and then we notice we can factor this: $x^2 - 5x + 6 = (x - 2)(x - 3)$, and so the solutions are $x = 2, x = 3$. 
(3) How many solutions does the equation \( \sin(x) + \cos(x) = 0 \) have in the interval \([0, 2\pi]\)?

(A) 1

(B) 2

(C) 3

(D) 4

\[ \text{ANSWER TO (3): } \boxed{B} \]

\[ \text{SOLUTION: } \] We want to solve \( \sin(x) = -\cos(x) \). Using the unit circle, we find that this occurs (for \( x \) in the interval \([0, 2\pi]\)) at

\[ x = \frac{3\pi}{4}, \quad x = \frac{7\pi}{4} \]

and so there are two solutions.

(4) What is the minimum value of the function \( \ln(x^2 + 1) + x^2 + 3 \)?

(A) \(-2\)

(B) \(-1\)

(C) 1

(D) 3

\[ \text{ANSWER TO (4): } \boxed{D} \]

\[ \text{SOLUTION: } \] Let’s first find the derivative of the function:

\[ \frac{d}{dx} \left( \ln(x^2 + 1) + x^2 + 3 \right) = \frac{d\ln(x^2 + 1)}{d(x^2 + 1)} \cdot \frac{d(x^2 + 1)}{dx} + \frac{d}{dx} (x^2 + 3) \]

\[ = \frac{1}{x^2 + 1} \cdot 2x + 2x \]

\[ = 2x \left( \frac{1}{x^2 + 1} + 1 \right) \]

and this is zero only at \( x = 0 \) (since the stuff in parenthesis is always bigger than zero). Thus, the minimum occurs at \( x = 0 \), but what is its value? To find this, plug \( x = 0 \) back into the function:

\[ \ln(0 + 1) + 0 + 3 = \ln(1) + 3 = 3 \]
(5) What is the derivative of \( \ln(\cos(x^2 - x)) - \sqrt{x + \sqrt{x}} + x \cos(x) \)

(A) \(-\frac{\sin(x^2 - x)(2x - 1)}{\cos(x^2 - x)} - \frac{1}{2} \left( 1 + \frac{1}{\sqrt{x}} \right) \frac{1}{\sqrt{x + \sqrt{x}}} + \cos(x) - x \sin(x) \)

(B) \(-\frac{\sin(x^2 - x)(2x - 1)}{\cos(x^2 + x)} - \frac{1}{2} \left( 1 - \frac{1}{\sqrt{x}} \right) \frac{1}{\sqrt{x + \sqrt{x}}} + \cos(x) - x \cos(x) \)

(C) \(-\frac{\sin(x^2 - x)(2x - 1)}{\cos(x^2 - x)} - \frac{1}{2} \left( 1 + \frac{1}{\sqrt{x}} \right) \frac{1}{\sqrt{x + \sqrt{x}}} + \cos(x) - x \sin(x) \)

(D) \(-\frac{\cos(x^2 - x)(2x - 1)}{\cos(x^2 - x)} - \frac{1}{2} \left( 1 + \frac{1}{\sqrt{x}} \right) \frac{1}{\sqrt{x + \sqrt{x}}} + \cos(x) - x \sin(x) \)

\textbf{ANSWER TO (5): A}

**SOLUTION:** There are three terms in the function; let’s find the derivative of each term:

\[
\frac{d}{dx} \left( \ln(\cos(x^2 - x)) \right) = \frac{1}{\cos(x^2 - x)} \cdot (\cos(x^2 - x))' = \frac{1}{\cos(x^2 - x)} \cdot (-\sin(x^2 - x)) \cdot (2x - 1) = -\frac{\sin(x^2 - x)(2x - 1)}{\cos(x^2 - x)}
\]

\[
\frac{d}{dx} \left( \sqrt{x + \sqrt{x}} \right) = \frac{1}{2} \left( x + \sqrt{x} \right)^{-1/2} \cdot (1 + \frac{1}{2\sqrt{x}}) = \frac{1}{2} \sqrt{x + \sqrt{x}} \cdot (1 + \frac{1}{2\sqrt{x}})
\]

\[
\frac{d}{dx} (x \cos(x)) = \cos(x) - x \sin(x)
\]

Putting these together, we see that A is the correct answer.

(6) The mass of a population of bacteria is expressed by the equation \( P(t) = ce^{kt} \). At time \( t = 0 \) the mass is 350 grams and is growing at the instantaneous growth rate of 14 grams per minute. What is its doubling time?
(A) $25 \ln(2) \text{ min}$

(B) $\frac{\ln(2)}{25} \text{ min}$

(C) $25 \ln\left(\frac{1}{2}\right) \text{ min}$

(D) $\frac{25}{\ln(2)} \text{ min}$

ANSWER TO (6): A

SOLUTION: We are given that $P(0) = c = 350$, so $P(t) = 350e^{kt}$. Also, we are given that the derivative of $P(t)$ at $t = 0$ is 14; that is,

$$P'(0) = 350ke^0 = 14 \implies k = \frac{14}{350} = \frac{1}{25}$$

and so $P(t) = 350e^{t/25}$. To find the doubling time,

$$2 \cdot 350 = 350e^{t/25} \implies 2 = e^{t/25} \implies \ln 2 = \frac{t}{25} \implies t = 25 \ln 2$$

(7) A bean casserole at room temperature ($25^\circ$) is put into an oven maintained at a constant temperature of $115^\circ$. If the temperature of the casserole is $55^\circ$ after 40 minutes, what temperature is the casserole when it’s taken out of the oven after a total of 2 hours?

(A) $78.3^\circ$

(B) $88.3^\circ$

(C) $82.3^\circ$

(D) $88.1^\circ$

ANSWER TO (7): B

SOLUTION: Let $T(t)$ be the temperature of the casserole, and $D(t) = 115 - T(t)$ the difference between the temperature of the oven and that of the casserole. The initial difference is $115 - 25 = 90$, and so

$$D(t) = 90 e^t$$

Now, at $t = 40$, we are given that $T = 55$, so $D(40) = 115 - 55 = 60$. But from the above formula for $D$, we have $D(40) = 90e^{40}$, and so

$$60 = 90e^{40} \implies e^{40} = \frac{2}{3} \implies c = (2/3)^{1/40}$$
which gives for $D$:

$$D(t) = 90 \left( \frac{2}{3} \right)^{\frac{t}{40}}$$

Plugging in $t = 120$ gives

$$D(120) = 90 \left( \frac{2}{3} \right)^{\frac{120}{40}}$$

$$= 90 \left( \frac{2}{3} \right)^3$$

$$= 90 \left( \frac{8}{27} \right)$$

and so

$$T(120) = 115 - 90 \frac{8}{27} = 88.3$$

(8) If the number of bears fishing in a stream is proportional to the number of salmon in that stream, and there are 8 bears fishing in a stream with 480 salmon, how many bears will be fishing in a stream with 720 salmon?

(A) 9

(B) 10

(C) 11

(D) 12

ANSWER TO (8): D

SOLUTION: For each bear, there are $\frac{480}{8} = 60$ salmon. So if there are 720 salmon, there will be $\frac{720}{60} = 12$ bears.

(9) Elmer has been diligently saving his loonies and twoonies and just opened a local bank account. What interest rate, compounded weekly, will triple the size of Elmer’s account in 2 years?

(A) 1.05 %

(B) 1.06 %
SOLUTION: We know that the amount of money $M(t)$ in the account after $t$ weeks, with an initial amount $M(0)$, will be

$$M(t) = M(0)(1 + \frac{r}{100})^t$$

for interest rate $r$ in percent. So we want to know, for what $r$ do we have

$$3M(0) = M(0)(1 + \frac{r}{100})^{104}$$

(because there are 104 weeks in 2 years). So

$$3 = (1 + \frac{r}{100})^{104}$$

$$3^{1/104} = 1 + \frac{r}{100}$$

$$r = 100(3^{1/104} - 1)$$

$$r = 1.06$$

(10) A population of bacteria grows proportional to its size. Its doubling time is four weeks. By what factor has it grown in 32 weeks?

(A) 512

(B) 248

(C) 256

(D) 128

SOLUTION: After 4 weeks, the initial population has doubled; after another four weeks, it has quadrupled. Since there are 8 4-week segments in 32 weeks, the population has grown by a factor of $2^8 = 256$. 

ANSWER TO (9): B

ANSWER TO (10): C
**Problem 2.** Your company has just extensively upgraded its website, making it much more appealing to your clientele. You launch the website first thing Monday morning and keep track of how many “hits” (that is, how many visits) the website receives on a given day. In the following table is the data for Monday, Tuesday, Thursday, and Friday; however no data was available for Wednesday. For example, on Thursday alone, your site received 189 hits.

<table>
<thead>
<tr>
<th>Day of the week</th>
<th>Number of visits</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monday</td>
<td>24</td>
</tr>
<tr>
<td>Tuesday</td>
<td>57</td>
</tr>
<tr>
<td>Thursday</td>
<td>189</td>
</tr>
<tr>
<td>Friday</td>
<td>288</td>
</tr>
</tbody>
</table>

(a) Plot the data points in the above table.
(b) Using a secant line, find an upper bound for the number of visits the website received on the Wednesday.

**SOLUTION:** Let’s use \( H(t) \) for the number of hits on day \( t \). We’ll number Monday as 1, Friday as 5 (though numbering Monday as 0, Friday as 4 is also possible; the answers will of course be the same).

The line connecting the data points for Tuesday and Thursday has slope
\[
m = \frac{189 - 57}{4 - 2} = 66
\]
so the equation of the line is
\[
H - 57 = 66(t - 2) \Rightarrow H = 66t - 75
\]
and plugging in \( t = 3 \) gives
\[
H = 123
\]
as an upper bound.

(c) Using a secant line, find a lower bound for the number of visits the website received on the Wednesday.

The line connecting the data points for Monday and Tuesday has slope
\[
m = \frac{57 - 24}{2 - 1} = 33
\]
so the equation of the line is
\[
H - 24 = 33(t - 1) \Rightarrow H = 33t - 9
\]
and plugging in \( t = 3 \) gives
\[
H = 90
\]
as a lower bound.

**OR**
The line connecting the data points for Thursday and Friday has slope \( m = \frac{288 - 189}{5 - 4} = 99 \) so the equation of the line is

\[ H - 189 = 99(t - 4) \Rightarrow H = 99t - 207 \]

and plugging in \( t = 3 \) again gives

\[ H = 90 \]

as a lower bound.
(d) On Tuesday, the number of hits is increasing at a rate of 44 hits/day. Find a new lower bound for the number of hits on the Wednesday.

The slope of the tangent at $t = 2$ is 44, so the tangent line has equation

$$H - 57 = 44(t - 2) \Rightarrow H = 44t - 31$$

and plugging in $t = 3$ gives

$$H = 101$$

as a lower bound.

(e) On Thursday, the number of hits is increasing at a rate of 88 hits/day. Find a new lower bound for the number of hits on the Wednesday.

The slope of the tangent at $t = 4$ is 88, so the tangent line has equation

$$H - 189 = 88(t - 4) \Rightarrow H = 88t - 163$$

and plugging in $t = 3$ again gives

$$H = 101$$

as a lower bound.
Problem 3. The copy service at King’s University is open for 5 and hours, from 9:00 to 14:00. Students are served in a first-come, first-served basis, and wait in line until it is their turn. Some very eager students arrive before 9:00. The graph below is a model for the number of students, $S(t)$, that have arrived at the Centre beginning at 9:00. The average time is to serve 50 students per hour.

(a) What is the length of the line at 11 o’clock?

The first thing we should do is draw the graph that represents people being served. Since every hour 50 students are served, this is a line of slope 50 starting at the origin (because people start being served as soon as the centre opens).

Now, 11 o’clock occurs at $t = 2$, because at $t = 0$ the time is 9 o’clock. The length of the line is the vertical distance between the two lines, which is seen to be $175 - 100 = 75$.

So, at 11:00, the line is 75 people long.
(b) When does the 150th student arrive in line?

We draw a line from 150 on the vertical axis (this represents the 150th person to arrive) to the graph, in order to find when this person arrives.

We see that they arrive at 1.5 hours past 9:00, so at 10:30.

(c) When does the student arrive that spends the longest time in line?

We need to maximize the horizontal distance between the two graphs; this maximum distance will be the longest waiting time. To do this, shift the straight line over, keeping it parallel, until it is tangent to the graph of $S(x)$. 
We see that the arrival time of the person who waits the longest is about 10:50.
(d) When does the line disappear?

The line disappears when the two graphs meet:

This happens at about 4.75 hours past 9, so at 1:45.

(e) If the arrivals graph is given by

\[ A(t) = (t - 6)^3 + 240 \]

find the exact value for the time in which the line is the longest.

We want to know for which values of \( t \) the slope of \( A(t) \) is 50:

\[ A'(t) = 50, \]

which gives

\[ 3(t - 6)^2 = 50 \]

solving for \( t \) one obtains

\[ t = 6 + \frac{5\sqrt{6}}{3}. \]
Problem 4. A cyclist decides to make a trip from Kingston to New York by pedaling for 100 Km a day. The amount of energy \( E \) that he spends per hour depends on his day average speed \( v \) as shown in the following graph.

(a) Describe the main properties of the graph: sign, growth and concavity. Explain these properties in the context of this problem.

The graph starts at zero, because it takes no energy to cycle at 0 km/hr! For any speed \( v \) other than zero, \( E \) is positive, because the cyclist is expending energy. The graph is increasing, because the faster he cycles, the more energy he’ll expend.

Furthermore, the rate at which \( E \) is increasing should get bigger for bigger \( v \); it doesn’t take a huge burst of energy to go from 8km/hour to 10 km/hr, but to go from 40 km/hr to 42 km/hr is a different story!
(b) Assume that the cyclist has a fixed expense of 250 cal/h due to his metabolism. Find the value of \( v \) which minimizes the total energy expended by the cyclist for the whole trip.

Since the cyclist is traveling a fixed distance, he wants to minimize the energy he burns each kilometre, given in cal/km.

If he cycles at a speed \( v \), every hour he expends \( E \) cal (where \( E \) is given by the graph), plus 250 cal due to metabolism. Thus

\[
\frac{\text{cal}}{\text{km}} = \frac{\text{cal/hr}}{\text{km/hr}} = \frac{E + 250}{v}
\]

and so

\[
\frac{E + 250}{v}
\]

is what he wants to minimize. This is interpreted on the graph as the slope of a line from the point \((0, -250)\) to a point on the graph, and we see that this slope is minimized when the line is tangent to the graph, at about \( v = 16 \text{km/hr} \).

(c) Answer question (b) by assuming that the cyclist decides to cover a distance of 120 km each day. No matter what distance he covers each day, he still expends 250 cal/hr, plus \( E \) cal from traveling at speed \( v \), according to the graph. How far he travels each day will affect how long he cycles each day (equivalently, how quick he gets there), but it does not affect his optimal speed!

The answer is the same, \( v = 16 \text{ km/hr} \).
(d) Answer question (b) by assuming that the cyclist fixed expense of energy is 500 cal/h.

The cyclist is still traveling a fixed distance, and so wants to minimize the energy he burns each kilometre, given in cal/km.

Now, if he cycles at a speed $v$, every hour he expends $E$ cal (where $E$ is given by the graph), plus 500 cal due to metabolism. Thus

$$\frac{\text{cal}}{\text{km}} = \frac{\text{cal/hr}}{\text{km/hr}} = \frac{E + 500}{v}$$

and so

$$\frac{E + 500}{v}$$

is what he wants to minimize. This is interpreted on the graph as the slope of a line from the point $(0, -500)$ to a point on the graph, and we see that this slope is minimized when the line is tangent to the graph, at about $v = 22.5 \text{km/hr}$.

(e) Find the **exact** value for question (b) assuming that

$$E(v) = \frac{24}{25} (v^2 + \frac{7}{4}v + 3).$$

The solution is given by the minimum of $(E(v) + 250)/v$, so that:

$$\frac{d}{dv} \left( \frac{E(v) + 250}{v} \right) = \frac{vE'(v) - (E(v) + 250)}{v^2},$$

hence we are looking for those values of $v$ for which

$$E(v) + 250 = vE'(v).$$
This last equation is satisfied when
\[ \frac{24}{25}(v^2 + \frac{7}{4}v + 3) + 250 = v \frac{24}{25}(2v + \frac{7}{4}) \]
which gives
\[ v^2 + \frac{7}{4}v + 3 + 250 \frac{25}{24} = 2v^2 + \frac{7}{4}v, \]
or equivalently
\[ v^2 = 250 \frac{25}{24} + 3. \]
Hence the solution is
\[ v = \frac{\sqrt{9483}}{6}. \]