1. [1pt each]
   (a) Prove that there are infinitely many primes \( p \) of the form \( 4n + 3 \) with \( n \in \mathbb{N} \).
   (b) For any natural number \( n > 1 \), \( n^3 - 4n \) is always divisible by 8 if \( n \) is even. What can you say if \( n \) is odd?

2. [1pt each] Let \( S = \{3n + 1 \mid n \in \mathbb{Z}, n \geq 0\} = \{1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, \ldots\} \). If \( a, b \in S \), we say that \( a \) is an \( S \)-divisor of \( b \) if \( b = ac \) for some \( c \in S \). An \( S \)-prime is an element \( 1 \neq q \in S \) whose only \( S \)-divisors are 1 and itself.
   (a) Prove that every element of \( S \) (except 1) is a product of \( S \)-primes.
   (b) Use the number 484 to show that factorization as a product of \( S \)-primes need not be unique.

3. [1pt each] Modular arithmetic can transform many questions that seem opaque into questions that are quite easy to answer. Consider the statement below:

   \( \text{If } p \geq 5 \text{ is a prime number, then } p^2 + 2 \text{ is always a composite number.} \)

Without looking ahead, think about how you would try and approach demonstrating this. Does it seem very easy?

Now:
   (a) List the squares mod 3.
   (b) Prove that if \( k \) is not a multiple of 3, then 3 divides \( k^2 + 2 \).
   (c) Prove that if \( p \) is a prime number and \( p \geq 5 \) then \( p^2 + 2 \) is a composite number.

4. [1pt each] Some more modular arithmetic.
   (a) Compute the remainders of 7, \( 7^2 \), \( 7^3 \), \( 7^4 \), \( 7^5 \), and \( 7^6 \) when divided by 29.
   (b) Compute the remainder of \( 7^{2017} \) when divided by 29.
   (c) Compute the remainders of 8, \( 8^2 \), \( 8^3 \), \( 8^4 \), \( 8^5 \), and \( 8^6 \) when divided by 29.
(d) Compute the remainder of $8^{2017}$ when divided by 29.

(e) Compute the remainder of $56^{2017}$ when divided by 29.

**Hints:**

(a) Finding the remainder when dividing by 29 is another way of talking about computing mod 29.

(b) Calculations mod 29 behave well with respect to arithmetic operations (multiplying, taking powers, etc).

(c) So you should definitely *not* be raising any number to the power 2017 and then computing the remainder.

5. [1pt each]

(a) Prove that there are no integers $a, b, c$ such that $a^2 + b^2 + c^2 = 999$.

(b) There are no perfect square integers of the form $3^m + 3^n + 1$ where $m$ and $n$ are positive integers.

(c) If $n \in \mathbb{Z}$, prove that $n^2 \equiv 0, 1, \text{ or } 4 \pmod{8}$.

6.

(a) [1pt] Prove that if $5 \nmid n - 1$, $5 \mid n$, $5 \nmid n + 1$, then $5 \mid n^2 + 1$.

(b) [2pts] Prove that if $2^p - 1$ is prime, then $p$ is prime. However, the converse is false. For instance, 11 is prime, but $2^{11} - 1$ is not prime.

(c) [2pts] Prove that if $p$ is a prime, $\sqrt{p}$ is irrational.

**Practice Problems:** Page 30 A.2(a)(b), 7, 8, B.11, 12, 16, 17, 21(a)(b), 22(a)(b). Page 36 A.2, 3, 6, 7. 9(a)(b), 11(a)(b)(c).