1. [1pt each]
   (a) If \( x_0, y_0 \) is a solution of the linear Diophantine equation \( ay - bx = k \), show that the general solutions are given by
   \[
   \begin{cases}
   x = x_0 + a't \\
   y = y_0 + b't
   \end{cases}
   \]
   where \( \gcd(a, b) = d \) and \( a = a'd \) and \( b = b'd \), and \( t \in \mathbb{Z} \).

   (b) Solve the following linear Diophantine equation
   \[ 299x + 481y = 78 \]

2. [1pt each] Solve the congruences
   (a) \( x^2 \equiv -3 \pmod{19} \).
   (b) \( x^3 + x^2 + 1 \equiv 0 \pmod{11} \).

3. [1pt each]
   Let \( n \in \mathbb{N} \) and \( a, b \in \mathbb{Z} \).
   (a) Prove that the linear congruence \( ax \equiv b \pmod{n} \) has a unique solution in \( \mathbb{Z}_n \) if \( \gcd(a, n) = 1 \).
   (b) Let \( n = 13 \), and take \( a = 3 \) and \( b = 5 \). Find the fraction \( 5/3 \pmod{13} \).
   (c) Let \( n = 13 \). Interpret the relation
   \[
   (1/2) + (2/3) = (7/6) \quad \text{in } \mathbb{Z}_{13}.
   \]

4. [1pt each]
   Find all solutions to the following equations:
   (a) \( 27x \equiv 1 \pmod{57} \)
   (b) \( 15x \equiv 9 \pmod{63} \)
5. [1pt each]

(a) If \( p \) is a prime number, and \( k < p \) is a positive integer, show that \( p \nmid k! \).

(b) Show that \( \binom{p}{k} \equiv 0 \pmod{p} \) for \( k = 1, \ldots, p - 1 \).

(c) If \( p \) is a prime number, prove that \( (x + y)^p \equiv x^p + y^p \pmod{p} \) for all integers \( x \) and \( y \).

(d) Let \( p \) be a prime number. Prove that \( x^p \equiv x \pmod{p} \) for all integers \( x \). (This is known as Fermat’s Little Theorem).

6.

(a) [2pts] Euler’s theorem: Let \( m \in \mathbb{N} \) and let \( \varphi(m) \) be the Euler function. Suppose that \( a \in \mathbb{Z} \) with \( \gcd(a, m) = 1 \). Then
\[
a^{\varphi(m)} \equiv 1 \pmod{m}.
\]

(b) [1pt] Find the smallest positive integer \( n \) that satisfies \( 3^{59} \equiv n \pmod{17} \)

(c) [1pt] The same question as (b): \( 7^{133} \equiv n \pmod{37} \)

7.

(a) [1pt] Find the continued fraction expansion for \( 61/48 \).

(b) [2pts] Using the continued fraction, find all solutions of the linear Diophantine equation
\[
61x - 48y = 1.
\]

Practice Problems: Page 36 B.14(b)(c), 16(c)(d). Page 41 A.1(c),2(b), B.11,14, 15(a)(b).