1. [1pt each] Find a solution of the congruence/the system of congruences:
   (a) \( 7x \equiv 8 \pmod{325} \).
   (b) \( x \equiv 3 \pmod{7}, x \equiv 4 \pmod{19}, x \equiv 10 \pmod{23} \).

2. [1pt each] Let \( M_2(\mathbb{Z}) \) be the set of all \( 2 \times 2 \) matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( a, b, c, d \in \mathbb{Z} \). Let \( M_2(\mathbb{Q}) \) be the set of all \( 2 \times 2 \) matrices with \( a, b, c, d \in \mathbb{Q} \).
   (a) Show that \( M_2(\mathbb{Z}) \) and \( M_2(\mathbb{Q}) \) are non-commutative rings with the identity.
   (b) Is \( M_2(\mathbb{Z}) \) a subring of \( M_2(\mathbb{Q}) \)?
   (c) Find all units of \( M_2(\mathbb{Z}) \), and all units of \( M_2(\mathbb{Q}) \).
   (d) Find all zero divisors of \( M_2(\mathbb{Z}) \), and find all zero divisors of \( M_2(\mathbb{Q}) \).

3. [1pt each] Let \( R \) be the set of all positive real numbers. Define a new addition \( \oplus \) and multiplication \( \otimes \) on \( R \) by
   \[ a \oplus b = ab, \quad a \otimes b = a^{\log b} \]
   (a) Is \( R \) a ring under these operations?
   (b) Is \( R \) a commutative ring?
   (c) Is \( R \) a field?

4. [1pt each]
   (a) Let \( S = \{0, 2, 4, 6, 8\} \) be a subset of \( \mathbb{Z}_{10} \). Is \( S \) a subring of \( \mathbb{Z}_{10} \)? Does \( S \) have the multiplicative identity?
   (b) Let \( \mathbb{Q}(\sqrt{3}) = \{ a + b\sqrt{3} \mid a, b \in \mathbb{Q} \} \). Is \( \mathbb{Q}(\sqrt{3}) \) a subfield of \( \mathbb{R} \)?
   (c) Let \( R \) be a ring with identity. If \( ab \) and \( a \) are units in \( R \), is \( b \) a unit in \( R \)?
   (d) If \( R \) and \( S \) are integral domains, then is \( R \times S \) an integral domain?
   (e) If \( R \) and \( S \) are fields, then is \( R \times S \) a field?
5. [1pt each]
   (a) Consider $\mathbb{R} \times \mathbb{R}$ with the usual addition and a new multiplication
   \[(a, b) \otimes (c, d) = (ac - bd, ad + bc)\]
   Show that $\mathbb{R} \times \mathbb{R}$ is a field.
   (b) Let $T$ be the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let $f, g \in T$ are given by
   \[f(x) = \begin{cases} 
   0 & \text{if } x \leq 2 \\
   x - 2 & \text{if } x > 2
   \end{cases}
   \]
   and
   \[g(x) = \begin{cases} 
   2 - x & \text{if } x \leq 2 \\
   0 & \text{if } x > 2
   \end{cases}
   \]
   Show that $T$ is not an integral domain.
   (c) Let $R$ be a ring such that $a^3 = a$ for every $a \in R$. Show that $R$ is commutative.
   (d) Let $R$ be a ring with identity. If there is a smallest positive integer $n$ such that $n1_R = 0$, then $R$ is said to have characteristic $n$. If no such $n$ exists, $R$ has characteristic zero. Show that $\mathbb{Z}$ has characteristic zero, and $\mathbb{Z}_n$ has characteristic $n$. What is the characteristic of $\mathbb{Z}_3 \times \mathbb{Z}_6$?

6. [1pt each]
   (a) Let
   \[R = \left\{ \left( \begin{array}{cc} a & -b \\ b & a \end{array} \right) | a, b \in \mathbb{R} \right\} .\]
   Define
   \[f : R \to \mathbb{C}, f \left( \left( \begin{array}{cc} a & -b \\ b & a \end{array} \right) \right) = a + b\sqrt{-1} .\]
   Show that $f$ is a ring homomorphism. Is it 1 − 1? or onto? Is $f$ an isomorphism?
   (b) Let $(\mathbb{Z}, \oplus, \odot)$ be the ring with a new addition and a new multiplication
   \[a \oplus b = a + b - 1, \ a \odot b = ab - (a + b) + 2 .\]
   Show that $(\mathbb{Z}, \oplus, \odot)$ is isomorphic to $(\mathbb{Z}, +, \times)$ (with the usual addition and multiplication).

**Practice Problems:** Page 448: 8, 12. Page 53: A 5,10,11,15,21; B.20,22,25,31,37.