The GCD-Criterion for Polynomials

Theorem 6 (GCD-criterion): Let $f_1, f_2, h \in F[x]$ be non-zero polynomials. Then there exist polynomials $k_1, k_2 \in f[x]$ such that

$$k_1 f_1 + k_2 f_2 = h \quad (1)$$

if and only if $\gcd(f_1, f_2) \mid h$.

Corollary 1: $\gcd(f, g) = 1 \Leftrightarrow$ there exist $h, k \in F[x]$ such that $fh + gk = 1$.

Corollary 2: If $h = \gcd(f, g)$, then $\gcd(\frac{f}{h}, \frac{g}{h}) = 1$.

Corollary 3: $\gcd(fh, gh) = \gcd(f, g) \cdot h$, if h is monic.

Corollary 4 (Euclid): If $f \mid gh$ and $\gcd(f, g) = 1$, then $f \mid h$.

Corollary 4′: If $f_i \mid g$ for $1 \leq i \leq n$ and $\gcd(f_i, f_j) = 1$ for all $i \neq j$, then $f_1 \cdot f_2 \cdots f_n \mid g$.

Corollary 5: If $\gcd(f, g) = 1$, then

$$\gcd(f, gh) = \gcd(f, h).$$