The Fundamental Theorem of Algebra

**Theorem A:** If \( f(x) \) is a polynomial of positive degree (i.e. \( \text{deg}(f) > 0 \)), then \( f \) has a root in \( \mathbb{C} \): there exists \( z \in \mathbb{C} \) such that \( f(z) = 0 \).

**Remarks:**
1) The first (correct) proof of this result was given by C.F. Gauss in 1797 (published 1799) when he was 20 (22) years old. He gave two more proofs in 1816, and a fourth in 1849.

2) Earlier proofs given by d’Alembert, Euler, Fontenex, Lagrange were criticized by Gauss.

3) The proofs are not easy; all use analysis.

**Corollary 1:** The only monic irreducible polynomials in \( \mathbb{C}[x] \) are the linear polynomials \( x - a \), \((a \in \mathbb{C})\).

**Corollary 2** ("Factorization Theorem in \( \mathbb{C}[x] \)"): Every polynomial \( f(x) \in \mathbb{C}[x] \) of positive degree has the factorization

\[
f(x) = c (x - a_1)^{n_1}(x - a_2)^{n_2} \cdots (x - a_r)^{n_r}.
\]

**Thus:** “Every polynomial \( f(x) \in \mathbb{C}[x] \) of degree \( n \) has precisely \( n \) roots in \( \mathbb{C} \), if we count the roots according to their multiplicities.”