The Lagrange Interpolation Formula

**Problem:** ("Exact Fit") Given the \( n \) "data points" \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\), where the \( x_i \)’s are distinct, find a polynomial of least degree such that

\[
\begin{align*}
  f(x_1) &= y_1 \\
  f(x_2) &= y_2 \\
  \vdots \quad \vdots \quad \vdots \\
  f(x_n) &= y_n
\end{align*}
\]

("graph of \( y = f(x) \) passes through \((x_1, y_1), \ldots, (x_n, y_n)\")

**Theorem 1:** ("Lagrange Interpolation Formula") The unique polynomial \( f(x) \) of degree \( \leq n - 1 \) which passes through \((x_1, y_1), \ldots, (x_n, y_n)\) is given by the formula

\[
f(x) = \sum_{k=1}^{n} y_k \frac{(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}
\]

\[
= \sum_{k=1}^{n} y_k \frac{g_k(x)}{g_k(x_k)},
\]

where

\[
g_k(x) = (x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)
\]

\[
= \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x-x_k)}.
\]
**Note:** There is a close connection between the Lagrange interpolation polynomial and remainders:

**Theorem 2:** Suppose

\[ g(x) = (x - a_1)(x - a_2) \cdots (x - a_n), \]

where the \( a_i \)’s are distinct. Then for any polynomial \( f(x) \) we have

\[
\text{rem}(f, g) = \sum_{k=1}^{n} f(a_k) e_k(x),
\]

where

\[
e_k(x) = \frac{g_k(x)}{g_k(a_k)} \quad \text{with} \quad g_k(x) = \frac{g(x)}{x - a_k}.
\]

Thus, \( \text{rem}(f, g) \) is the Lagrange interpolation polynomial of the data points \((a_1, f(a_1)), (a_2, f(a_2)), \ldots, (a_n, f(a_n))\).

**Remarks.** Note that the \( e_k \)’s depend only on \( g \) (and not on \( f \)); we call these the constituent polynomials of \( g \).