Fourier Approximation

Problem C: Given: 1) A function \( g \in C[a, b] \), and 2) “simple” functions \( f_1, \ldots, f_k \in C[a, b] \).

Find: a linear combination

\[ f_0 = a_1 f_1 + \ldots + a_k f_k \]

which best approximates the function \( g \) in the \( L^2 \)-norm; in other words, find \( f_0 \) as above such that

\[ \int_a^b (f_0 - g)^2 dx \leq \int_a^b (f - g)^2, \]

for all functions \( f = a'_1 f_1 + \ldots + a'_n f_n \).

Solution: Step 1: Apply the Gram-Schmidt Method to the functions \( f_1, \ldots, f_k \) by defining the dot product of two functions \( f, g \in C[a, b] \) to be

\[ (f \cdot g) = \int_a^b f(x)g(x)dx. \]

This gives us orthogonal functions \( h_1, \ldots, h_k \) which are certain linear combinations of \( f_1, \ldots, f_k \).

Step 2: Use the projection formula of Theorem 9:

\[ f_0 = PV(g) = \sum_{i=1}^k \frac{(g \cdot h_i)}{(h_i \cdot h_i)} h_i \]