Difference Equations as Discrete Linear Systems

Definition: A difference equation of order k is a sequence of equations of the form

$$(1)\quad a_{n+k} = c_0 a_n + c_1 a_{n+1} + \ldots + c_{k-1} a_{n+k-1} + b_n, \qquad n \geq 0,$$

where $c_0, c_1, \ldots, c_{k-1}$ and $b_0, b_1, \ldots, b_n, \ldots$ are given and we want to solve for the a_n's.

Theorem 14: Given the difference equation (1), put

$$\vec{u}_n = \begin{pmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & \cdots & c_{k-2} & c_{k-1} \end{pmatrix},$$

and $\vec{b}_n = (0, \ldots, 0, b_n)^t$. Then we have the discrete linear system

$$(2)\qquad \vec{u}_{n+1} = A\vec{u}_n + \vec{b}_n.$$

Remark: The above matrix A has characteristic polynomial

$$\mathrm{ch}_A(t) = t^k - c_{k-1}t^{k-1} - \ldots - c_1 t - c_0.$$