Generalized Eigenvectors

**Definition:** Let \( A \) be an \( m \times m \) matrix and \( \lambda \in \mathbb{C} \). A vector \( \vec{v} \in \mathbb{C}^m \) is called a *generalized \( \lambda \)-eigenvector* of \( A \) *of order* \( \leq p \) if
\[
(A - \lambda I)^p \vec{v} = \vec{0}.
\]
If, in addition, \((A - \lambda I)^{p-1} \vec{v} \neq \vec{0}\), then we say that \( \vec{v} \) has (precise) order (or degree) \( p \). The space
\[
E^p_A(\lambda) = \text{Nullsp}((A - \lambda I)^p)
\]
\[
= \{ \vec{v} \in \mathbb{C}^m : (A - \lambda I)^p \vec{v} = \vec{0} \}.
\]
is called the \( p \)-th *generalized \( \lambda \)-eigenspace* of \( A \), and its dimension
\[
\nu^p_A(\lambda) := \dim E^p_A(\lambda) = m - \text{rank}((A - \lambda I)^p)
\]
is the \( p \)-th *generalized geometric multiplicity* of \( A \) with respect to \( \lambda \).

**Remarks:**
1) For \( p = 1 \) we recover the usual eigenspace:
\[
E^1_A(\lambda) = E_A(\lambda), \quad \text{and so } \nu^1_A(\lambda) = \nu_A(\lambda).
\]
2) It is clear that \( E^p_A(\lambda) \subset E^{p+1}_A(\lambda), \forall p \). Thus
\[
\{0\} \subset E^1_A(\lambda) = E_A(\lambda) \subset E^2_A(\lambda) \subset \ldots E^p_A(\lambda) \subset \ldots
\]
Example: If \( A = J(\lambda, m) \) is a Jordan block, then \( E^p_{J(\lambda,m)} = \text{Nullsp}((A - \lambda I)^p) = \langle \vec{e}_1, \ldots, \vec{e}_p \rangle \), if \( p \leq m \), and \( E^p_{J(\lambda,m)} = \mathbb{C}^m \), if \( p \geq m \). Thus
\[
\nu^p_{J(\lambda,m)}(\lambda) = \begin{cases} 
p & \text{if } p \leq m, \\
m & \text{if } p \geq m. 
\end{cases}
\]

Theorem 5 (Properties of gen. eigenvectors):
(a) If \( A = \text{Diag}(B, C) \) and if \( p \geq 1 \), then
\[ E^p_A(\lambda) = E^p_A(\lambda) \oplus E^p_B(\lambda); \]
in particular, \( \nu^p_A(\lambda) = \nu^p_B(\lambda) + \nu^p_C(\lambda) \).
(b) If \( A = PBP^{-1} \), then
\[ E^q_A(\lambda) = PE^q_B(\lambda), \quad \text{for all } q \geq 1; \]
in particular, \( \nu^q_A(\lambda) = \nu^q_B(\lambda) \), for all \( q \geq 1 \).
(c) If \( J \) is a Jordan canonical form of \( A \), then
\[ \nu^p_A(\lambda) - \nu^{p-1}_A(\lambda) = \#(\text{Jordan blocks } J(\lambda, k_{ij}) \text{ of JCF } J \text{ of size } k_{ij} \geq p). \]
(d) We have that
\[ \nu^p_A(\lambda) = \nu^{p+1}_A(\lambda) \Rightarrow \nu^p_A(\lambda) = \nu^{p+q}_A(\lambda), \quad \text{for all } q \geq 1. \]

Corollary: The numbers \( \nu^p_A(\lambda_i) \) determine the JCF of \( A \) (by taking second differences).
Example 1: Let $A$ be a matrix with characteristic polynomial $\text{ch}_A(t) = (t - 7)^5$ and generalized geometric multiplicities

$$\nu^*_A(7) = (\nu^1_A(7), \nu^2_A(7), \nu^3_A(7), \ldots) = (2, 4, 5, 5, 5, \ldots)$$

Find the associated Jordan canonical form $J$ of $A$.

Solution: By using Theorem 5 (or its Corollary) we obtain:

\[
\begin{align*}
\nu^4_A - \nu^3_A &= 5 - 5 = 0 \Rightarrow 0 \text{ blocks of size } \geq 4 \\
\nu^3_A - \nu^2_A &= 5 - 4 = 1 \Rightarrow 1 \text{ block of size } \geq 3 \\
\nu^2_A - \nu^1_A &= 4 - 2 = 2 \Rightarrow 2 \text{ blocks of size } \geq 2 \\
\nu^1_A - \nu^0_A &= 2 - 0 = 2 \Rightarrow 2 \text{ blocks of size } \geq 1
\end{align*}
\]

and hence $J$ has: 0 blocks of size $1 \times 1$
1 block of size $2 \times 2$
1 block of size $3 \times 3$
0 blocks of size $4 \times 4$ etc.

Thus: $J = \text{Diag}(J(7, 3), J(7, 2)) =
\begin{pmatrix}
7 & 1 & 0 & 0 & 0 \\
0 & 7 & 1 & 0 & 0 \\
0 & 0 & 7 & 0 & 0 \\
0 & 0 & 0 & 7 & 1 \\
0 & 0 & 0 & 0 & 7
\end{pmatrix}$

(up to order of the blocks).
**Example 2:** Find the Jordan canonical form of

\[
A = \begin{pmatrix} 7 & 0 & 0 & 1 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 1 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 7 \end{pmatrix}.
\]

**Solution:**

**Step 1:** Find the characteristic polynomial.

Expanding \(\det(A - tI)\) along the first row yields

\[\text{ch}_A(t) = (-1)^5 \det(A - tI) = (t - 7)^5.\]

Thus, \(m_A(7) = 5\). Moreover, since

\[B = A - 7I = (\vec{e}_3|\vec{e}_5|\vec{0}|\vec{e}_1|\vec{0}),\]

we have \(\nu_A(7) = 5 - \text{rk}(B) = 2 = \#(\text{Jordan blocks})\).

**Step 2:** Calculate the generalized geometric multiplicities.

Since \(B(\vec{v}_1|\vec{v}_2|\vec{v}_3|\vec{v}_4|\vec{v}_5) = (B\vec{v}_1|B\vec{v}_2|B\vec{v}_3|B\vec{v}_4|B\vec{v}_5)\)
and since \(B\vec{e}_i = i\)-th column of \(B\), we see that

\[B^2 = (B\vec{e}_3|B\vec{e}_5|B\vec{0}|B\vec{e}_1|B\vec{0}) = (\vec{0}|\vec{0}|\vec{0}|\vec{e}_3|\vec{0}),\]

and similarly

\[B^3 = B \cdot B^2 = (B\vec{0}|B\vec{0}|B\vec{0}|B\vec{e}_3|B\vec{0}) = (\vec{0}|\vec{0}|\vec{0}|\vec{0}|\vec{0}),\]

and so \(B, B^2, \text{ and } B^3\) have ranks 3, 1, and 0 respectively. Moreover, clearly \(B^p = 0\), for all \(p \geq 3\).
Thus, since $\nu^p_A(7) = 5 - \text{rank}(B^p)$, we see that the
generalized geometric multiplicities are

$$\nu^*_A(7) = (2, 4, 5, 5, \ldots).$$

**Step 3:** Find the Jordan blocks by the method of second differences.

Since $\text{ch}_A(t) = (t - 7)^5$ and since the generalized geometric multiplicities are

$$\nu^*_A(7) = (2, 4, 5, 5, \ldots),$$

we can conclude by Example 1 that the Jordan canonical form $J$ of $A$ is

$$J = \text{diag}(J(7, 3), J(7, 2)) = \begin{pmatrix} 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$