Geometric Series of Matrices

**Definition:** Let $T$ be any square matrix. Then the sequence $\{S_n\}_{n \geq 0}$ defined by

$$S_n = I + T + \ldots + T^{n-1}, \quad S_0 = I,$$

is called the geometric series generated by $T$. The series converges if the sequence $\{S_n\}_{n \geq 0}$ converges; we then write

$$\sum_{n=0}^{\infty} T^n = \lim_{n \to \infty} S_n.$$

**Theorem 11:** The geometric series generated by $T$ converges if and only if

\[(1) \quad |\lambda_i| < 1, \text{ for each eigenvalue } \lambda_i \text{ of } T.\]

If this condition holds, then $I - T$ is invertible and we have

\[(2) \quad S_n := \sum_{k=0}^{n-1} T^k = (I - T)^{-1}(I - T^n),\]

and hence the series converges to

\[(3) \quad \sum_{k=0}^{\infty} T^k = (I - T)^{-1}.\]
Remark: Note that condition (1) is equivalent to:

\[ \lim_{{n \to \infty}} T^n = 0. \]