Stochastic Matrices and Markov Chains

Definition: A real \( p \times q \) matrix \( A = (a_{ij}) \) is called stochastic if

(1) \( a_{ij} \geq 0 \), for all \( i, j \);

(2) \( \sum_p A = \sum_q \),

where \( \sum_p = (1, 1, \ldots, 1) \in \mathbb{R}^p \).

A (homogeneous) Markov chain is a discrete linear system

\[ \vec{v}_{n+1} = A\vec{v}_n \]

in which \( \vec{v}_n \) is a stochastic vector, and \( A \) is a stochastic matrix.

Remarks: 1) Condition (2) means: the sum of each column of \( A \) is equal to 1.

2) Markov chains were named after A.A. Markov (1856 – 1922) by A.N. Kolmogorov in 1935.

Properties: 1) \( A, B \) stochastic \( \Rightarrow AB \) stochastic.

2) If \( A \) is a square stochastic matrix, then
   a) \( \lambda_1 = 1 \) is a regular eigenvalue of \( A \);
   b) \( |\lambda_i| \leq 1 \) for every eigenvalue \( \lambda_i \) of \( A \).