Math 211
Assignment 3

Due 2 November 2015

Do the following problems by hand (and show your work).

1. Verify that the following two identities hold for any \( n \geq 1 \):
   \[
   \sum_{k=0}^{n} \binom{n}{k} = 2^n \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.
   \]
   Hint: Use \((x + y)^n\).

2. Find the following remainders (by hand). Explain your method by writing down your intermediate calculations.
   (a) \( \text{rem}(2^{12}, 63) \).
   (b) \( \text{rem}(8^5 + 9^5, 17) \).
   (c) \( \text{rem}(3^8 - 4^8, 10) \).
   (d) \( \text{rem}(24 \cdot 25 + 27 \cdot 29, 26) \).
   (e) \( \text{rem}(103 \cdot 65 + 329 \cdot 663, 33) \).

3. Use modulo 9 calculations to show that
   \[ 234785346 \cdot 5683592187 \neq 1334424157147691702. \]
   Note: Recall from class that if \( a = b \), then we know \( a \equiv b \pmod{n} \) for any \( n \in \mathbb{N} \), in particular \( a \equiv b \pmod{9} \).

4. Use modular arithmetic to find the following remainders by hand.
   \[
   \begin{align*}
   (a) \ & \text{rem}(5^{18}, 11); \\
   (b) \ & \text{rem}(7^{14}, 18).
   \end{align*}
   \]

5. Prove that every positive integer is congruent to the sum of its digits modulo 3.
   Hint: First, write \( n \) as
   \[ n = c_r 10^r + c_{r-1} 10^{r-1} + \ldots + c_1 10 + c_0, \]
   where \( 0 \leq c_i \leq 9 \) \((0 \leq i \leq r)\). (All I mean here is, for example, \( 4536 = 4 \cdot 10^3 + 5 \cdot 10^2 + 3 \cdot 10^1 + 6 \).)
M1.[Optional. Not graded] (a) Use the MAPLE command \texttt{isolve(.)} to find the general integer solution of the Diophantine equation $8023x + 8249y = 1243$. Interpret your MAPLE output and compare it to the solution we obtained in Question 1(a) of Assignment #2.

(b) Use the MAPLE commands \texttt{ifactor(.)} and \texttt{ifactors(.)} to find the prime factorization of the number 123456789. Explain your output and comment on the similarity and difference between the two results obtained.

(c) Use the commands \texttt{ithprime(.)} and \texttt{[seq(.)]} to write a program \texttt{firstprimes(n)} which returns the list of the first $n$ primes. Use it to find the first 20 and also the first 50 primes.