Math 211

Assignment 1 - Solutions

1. (a) By the Euclidean algorithm, we have

\[
\begin{align*}
2235 &= 1 \cdot 1245 + 990 \\
1245 &= 1 \cdot 990 + 255 \\
990 &= 3 \cdot 225 + 30 \\
225 &= 7 \cdot 30 + 15 \\
990 &= 3 \cdot 255 + 225 \\
225 &= 7 \cdot 30 + 15
\end{align*}
\]

Thus, \( \gcd(1245, 2235) = 15 \).

(b) Since \( 1245 \div 15 = 83 \) and \( 2235 \div 15 = 149 \), we see that

\[
\frac{1245}{2235} = \frac{83}{149},
\]

which is in lowest terms.

2. (a) By the Euclidean algorithm, we have

\[
\begin{align*}
41449 &= 1 \cdot 39617 + 1832 \\
39617 &= 21 \cdot 1832 + 1145 \\
1832 &= 1 \cdot 1145 + 687 \\
1145 &= 1 \cdot 687 + 458 \\
687 &= 1 \cdot 458 + 229 \\
458 &= 2 \cdot 229
\end{align*}
\]

Thus, \( \gcd(41449, 39617) = 229 \).

(b) By the Euclidean algorithm, we have

\[
\begin{align*}
22307 &= 1 \cdot 12553 + 9754 \\
12553 &= 1 \cdot 9754 + 2799 \\
9754 &= 3 \cdot 2799 + 1357 \\
2799 &= 2 \cdot 1357 + 85 \\
1357 &= 16 \cdot 85 + 1
\end{align*}
\]

Thus, \( \gcd(22307, 12553) = 1 \).

3. By the division algorithm we have \( 1000 = 76 \cdot 13 + 12 \). Thus, dividing both sides by 13 we obtain

\[
\frac{1000}{13} = 76 + \frac{12}{13},
\]

and so \( \frac{1000}{13} = 76 + \frac{12}{13} \).

4. Since \( 2000 = 117 \cdot 17 + 11 \), there are 117 numbers between 1 and 2000 which are divisible by 17. Of these, 35 are less than 600 (because \( 600 = 35 \cdot 17 + 5 \)), and so there are \( 82 = 117 - 35 \) numbers between 2000 and 600 which are multiples of 17.

5. Let \( P(n) \) be the statement: \( \text{“} n^3 - n \text{ is divisible by 6.”} \) Then the statement \( P(1) \) is true since \( 6|0 = 1^3 - 1 \). Now, suppose \( P(k) \) is true for some \( k \in \mathbb{N} \), i.e. that \( 6|(k^3 - k) \); we want to show that \( P(k+1) \) is also true. First we simplify the \( (k + 1)^3 - (k + 1) \) as follows,

\[
(k + 1)^3 - (k + 1) = (k^3 + 3k^2 + 3k + 1) - (k + 1) = k^3 + 3k^2 + 2k = (k^3 - k) + 3(k^2 + k).
\]

Given any \( k \in \mathbb{N} \), one of \( k \) or \( k + 1 \) must be even; therefore, \( k^2 + k = k(k + 1) \) is even, so we can write \( k^2 + k = 2h \) for some \( h \in \mathbb{N} \). Thus, we can simplify the above expression, and obtain

\[
(k + 1)^3 - (k + 1) = k^3 - k + 6h.
\]

By the induction hypothesis (i.e. that we are assuming \( P(k) \) is true), we have \( 6|(k^3 - k) \), and clearly we have \( 6|(6h) \). Thus by the divisibility property, we obtain \( 6|(k^3 - k + 6h) = ((k + 1)^3 - (k + 1)) \), which means that \( P(k+1) \) is true. Therefore, we obtain by induction that \( P(n) \) is true for every positive integer \( n \geq 0 \).

6. (a) Clearly \( 2|294 \) and \( 2|816 \), but \( 2 \) does not divide \( -9 \). Thus, \( 2 \nmid \gcd(294, 816) \) but \( 2 \nmid (-9) \) and hence the GCD-criterion shows that the equation cannot have any integral solutions.

(b) The Euclidean algorithm yields

\[
1628 = 2 \cdot 777 + 44,
\]

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777 = \cdot 10 \cdot 74 + 37,
\quad and
74 = 2 \cdot 37 + 0,
which means that gcd(1628, 777) = 37. Since 37|37, we know by the GCD-criterion that the equation has an integer solution. By back-substitution, we obtain

\quad 37 = 777 - \cdot 10 \cdot 74 = 777 - 10(1628 - 2 \cdot 777) = 21 \cdot 777 + (-10) \cdot 1628.

Then, we can deduce that \( x = 21 \) and \( y = 10 \) is a solution to the equation \( 777x - 1628y = 37 \).

(c) Here the Euclidean algorithm gives

\[
\begin{array}{c|c}
2115 & = 1 \cdot 1233 + 882 \\
1233 & = 1 \cdot 882 + 351 \\
882 & = 2 \cdot 351 + 180 \\
351 & = 1 \cdot 180 + 171 \\
180 & = 1 \cdot 171 + 9 \\
171 & = 19 \cdot 9 \\
\end{array}
\]

Thus, gcd(2115, 1233) = 9. Since, 9|18, we know by the GCD-criterion that the equation has an integral solution. By back-substitution we obtain

\[
9 = 180 - 171 = 180 - 3(51) = 2 \cdot 180 - 3 \cdot 51 = 2 \cdot 351 + 2 \cdot 882 - 5 \cdot 1233 - 3 \cdot 882 = 7 \cdot 882 - 5 \cdot 1233 = 7(2115 - 1233) - 5 \cdot 1233 = 7 \cdot 2115 - 12 \cdot 1233.
\]

Thus, \( x_0 = -12 \), \( y_0 = 7 \) is a solution of the equation \( 1233x_0 + 2115y_0 = 9 \), and so \( x = -24 \), \( y = 14 \) is a solution of the equation \( 1233x + 2115y = 18 \).

7. (a) We have

\[
x^2 + y^2 = (t^2 - s^2)^2 + (2st)^2 = t^4 - 2t^2s^2 + s^4 + 4s^2t^2 = t^4 + 2t^2s^2 + s^4 = (t^2 + s^2)^2 = z^2
\]
and so \( x^2 + y^2 = z^2 \), as claimed.

(b) For \( t = s + 1 \) we have \( x = t^2 - s^2 = (s^2 + 2s + 1) - s^2 = 2s + 1 \), \( y = 2st = 2s(s + 1) \), and \( z = t^2 + s^2 = (s^2 + 2s + 1) + s^2 = 2s^2 + 2s + 1 = 2(s + 1) + 1 \). Thus, \((x, y, z) = (2s + 1, 2s(s + 1), 2(s + 1) + 1)\). For \( s = 1 \) we get \((x, y, z) = (2)(1) + 1, 2(1)(1 + 1), 2(1)(1 + 1) + 1 = (3, 4, 5)\). Similarly, for \( s = 2 \) we obtain \((x, y, z) = (2)(2) + 1, 2(2)(3), 2(2)(3) + 1 = (5, 12, 13)\), for \( s = 3 \) we get \((x, y, z) = (2)(3) + 1, 2(3)(4), 2(3)(4) + 1 = (7, 24, 25)\), and for \( s = 4 \) we have \((x, y, z) = (2)(4) + 1, 2(4)(5), 2(4)(5) + 1 = (9, 40, 41)\).

8. (a) Let \( x = 1 \) and \( y = -1 \). Then we have \( x | y \) and \( y | x \), but \( x \neq y \).

(b) Let \( a = 1 \), \( b = 2 \) and \( d = 4 \). Then we have \( a | d \) and \( b | d \), but we can easily find \( x, y \in \mathbb{Z} \) such that \( ax + by \nmid d \). For example, if we take \( x = y = 100 \), then we have

\[
ax + by = 1 \cdot 100 + 2 \cdot 100 = 300 \nmid 4 = d.
\]

B1. Let \( a, b, c \in \mathbb{N} \) be such that \( a | (b + c) \) and \( gcd(b, c) = 1 \). Prove that \( gcd(a, b) = 1 \).

Proof: We prove by contradiction. Suppose \( gcd(a, b) = d > 1 \). Then by the definition of \( gcd \), \( d \) divides \( a \) and \( b \). Then, since \( d | a \) and \( a | (b + c) \), it follows by one of the divisibility properties that \( d | (b + c) \). Because \( d \) \mid b \) and \( d | (b + c) \), it follows that \( d \) divides \(-1) \cdot b + 1 \cdot (b + c) = c \), i.e. \( d | c \). Thus we have \( d | b \) and \( d | c \), from which it follows that \( gcd(b, c) \geq d > 1 \), and this is a contradiction. Therefore, \( gcd(a, b) \) can not be greater than 1, i.e. it must be 1.