Math 211

Assignment 6 - Solutions

1. Write \( f = q(cg) + r \), where \( q, r \in F[x] \) and \( \deg(r) < \deg(cg) \). Thus \( \text{quot}(f, cg) = q \) and \( \text{rem}(f, cg) = r \). Since we also have that \( f = (qc)g + r \) and that \( \deg(r) < \deg(cg) = \deg(g) \), it follows that \( \text{quot}(f, g) = qc \) and \( \text{rem}(f, g) = r = \text{rem}(f, g) \), as desired.

2. If \( f = g \) then clearly \( f | g \) because \( g = f \cdot 1 \). Conversely, suppose that \( f | g \), so \( g = fh \), for some \( h \in R[x] \). Since \( \deg(h) = \deg(g) - \deg(f) = 0 \), it follows that \( h(x) = c \) is constant. Thus \( cf = g \) and hence \( c = 1 \) because \( f \) and \( g \) are both monic. This proves \( f = g \).

3. (a) We have by the division algorithm

\[
x^{1999} + x = q(x)(2x + 1) + r(x),
\]

where \( \deg r < 1 \). Thus \( r(x) = c \) for some \( c \in \mathbb{Q} \). Substituting \( x = -1/2 \), we obtain

\[
c = (-\frac{1}{2})^{1999} + (-\frac{1}{2}) = -\frac{1}{2}^{1999} - \frac{1}{2} = -2^{1998} + 1.
\]

Therefore,

\[
\text{rem}(x^{1999} + x, x - (\frac{1}{2})) = c = -2^{1998} + 1.
\]

Note: Do not give approximate answers! (In particular, \((\frac{1}{2})^{1999} \neq 0\) but is only close to 0.)

(b) Put \( f(x) = x^{1999} + x \) and \( g(x) = x^2 - 3x + 2 = (x - 1)(x - 2) \). Since \( \deg(g) = 2 \), \( r(x) := \text{rem}(f, g) \) has degree \( \leq 1 \), so \( r(x) = r_0 + r_1 x \), for some numbers \( r_0, r_1 \). To determine \( r_0 \) and \( r_1 \), we use the fact (proven in class) that \( r(a_i) = f(a_i) \), where \( a_1 = 1 \) and \( a_2 = 2 \) are the roots of \( g(x) \). Thus

\[
\begin{align*}
r_0 + 1 \cdot r_1 &= r(1) = f(1) = 1^{1999} + 1 = 2 & \text{or} & & r_0 + 1 \cdot r_1 = 2 \\
r_0 + 2 \cdot r_1 &= r(2) = f(2) = 2^{1999} + 2 \\
\end{align*}
\]

Subtracting the first equation from the second yields \( r_1 = 2^{1999} \), and so from the first equation we obtain \( r_0 = 2 - r_1 = 2 - 2^{1999} \). Thus, the remainder is \( r(x) = 2^{1999}x + 2 - 2^{1999} \).

(c) We know \( \text{rem}(x^{1999} + x, x + \omega) = (-\omega)^{1999} + (-\omega) \). We now simplify this expression. Put \( \omega = -\frac{1+i\sqrt{3}}{2} = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}) \). Then by de Moivre’s formula \( \omega^3 = 1 \), because \( \omega^3 = \cos(3\frac{2\pi}{3}) + i\sin(3\frac{2\pi}{3}) = \cos(2\pi) + i\sin(2\pi) = \cos(0) + i\sin(0) = 1 \). Therefore, \( \omega^{1999} = (\omega^3)^{666} \omega = \omega \). Thus \( \text{rem}(x^{1999} + x, x + \omega) = (-\omega)^{1999} + (-\omega) = -\omega - \omega = 0 - i\sqrt{3} \).

(d) By the quadratic formula we have \( x^2 + x + 1 = (x - \omega)(x - \overline{\omega}) \), where, as before, \( \omega = -\frac{1+i\sqrt{3}}{2} \). Thus \( \omega^3 = 1 \). As in part (b) we have that \( r(x) = \text{rem}(f, x^2 + x + 1) = r_0 + r_1 x \), and that \( f(x) = r(x) \), and that \( f(\overline{\omega}) = r(\overline{\omega}) \). This leads to the system of equations

\[
\begin{align*}
r_0 + r_1 \omega &= \omega^{1999} + \omega = 2 \omega \\
r_0 + r_1 \overline{\omega} &= \overline{\omega^{1999}} \overline{\omega} = 2 \overline{\omega},
\end{align*}
\]

where we have used the fact that \( \omega^{1999} = \omega \) (cf. part(c)). Solving this system yields \( r_1 = 2, r_0 = 0 \), and so \( \text{rem}(x^{1999} + x, x^2 + x + 1) = r_1 x + r_0 = 2x \).

4. (a) The Euclidean algorithm yields:

\[
\begin{align*}
4x^4 + x^2 - x - 1 &= (x - 1)(4x^3 + 4x^2 + x) + 4x^2 - 1 \\
4x^3 + 4x^2 + x &= (x + 1)(4x^2 - 1) + 2x + 1 \\
4x^2 - 1 &= (2x - 1)(2x + 1)
\end{align*}
\]

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Thus, the \( \gcd(4x^4 + x^2 - x - 1, 4x^3 + 4x^2 + x) = c(2x + 1) \) for some \( c \in \mathbb{Q} \). Since the \( \gcd \) was defined to be monic, this forces \( 2c = 1 \) or \( c = \frac{1}{2} \), and so \( \gcd(f_1, f_2) = x + \frac{1}{2} \).

(b) By back-substitution we obtain

\[
2x + 1 = (4x^3 + 4x^2 + x) - (x + 1)(4x^2 - 1)
= (4x^3 + 4x^2 + x) - (x + 1)((4x^4 + x^2 - x - 1) - (x - 1)(4x^3 + 4x^2 + x))
= (1 + (x - 1)(x + 1))(4x^3 + 4x^2 + x) - (x + 1)(4x^4 + x^2 - x - 1)
= (x^2)(4x^3 + 4x^2 + x) - (x + 1)(4x^4 + x^2 - x - 1).
\]

Thus, \( x + \frac{1}{2} = (\frac{1}{2}x^2)(4x^3 + 4x^2 + x) - \frac{1}{2}(x + 1)(4x^4 + x^2 - x - 1) = (\frac{1}{2}x^2)f_2(x) - \frac{1}{2}(x + 1)f_1(x) \), and so the desired polynomials are

\[
a(x) = -\frac{1}{2}(x + 1) \quad \text{and} \quad b(x) = \frac{1}{2}x^2.
\]

5. (a) Since \( b^2 - 4c = (-1)^2 - 4 \cdot 2 = -7 < 0 \), and there does not exist \( \alpha \in \mathbb{Q} \) such that \( \alpha^2 = -7 \), the polynomial \( f(x) = x^2 - x + 2 \) is irreducible over \( \mathbb{Q} \) by Theorem 3.8.

(b) We know by the quadratic formula that

\[
x^2 - x + 2 = (x - \frac{1 + i\sqrt{7}}{2})(x - \frac{1 - i\sqrt{7}}{2}).
\]

Since \( (x - \frac{1 + i\sqrt{7}}{2}) \) and \( (x - \frac{1 - i\sqrt{7}}{2}) \) are in \( \mathbb{C}[x] \) and of positive degree, we see that \( x^2 - x + 2 \) is NOT irreducible over \( \mathbb{C} \).

6. (a) Since \( x^2 + x - 2 = (x - 1)(x + 2) \) and \( x^2 + x + 1 = (x - \omega)(x - \overline{\omega}) \), with \( \omega = \frac{-1 + i\sqrt{7}}{2} \), it follows that the factorization of \( f \) in \( \mathbb{C}[x] \) is given by:

\[
\tag{1} f(x) = (x - 1)^5(x + 2)^2(x - \omega)^3(x - \overline{\omega})^3.
\]

Thus, the roots of \( f \) are 1, \(-2\), \(\omega\), \(\overline{\omega}\) with the following multiplicities:

\[
\text{mult}_1(f) = 5, \quad \text{mult}_{-2}(f) = 2, \quad \text{mult}_{\omega}(f) = \text{mult}_{\overline{\omega}}(f) = 3.
\]

Similarly, \( g(x) \) factors as

\[
\tag{2} g(x) = (x + 2)^2(x - 2)^3(x - 1)^3(x - \omega)^4(x - \overline{\omega})^4,
\]

and the roots of \( g \) are \(-2, 2, 1, \omega, \overline{\omega}\) with multiplicities

\[
\text{mult}_{-2}(g) = 2, \quad \text{mult}_2(g) = \text{mult}_1(g) = 3, \quad \text{mult}_{\omega}(g) = \text{mult}_{\overline{\omega}}(g) = 4.
\]

(b) We have by part (a)

\[
\tag{3} f(x) = (x - 1)^5(x + 2)^2(x - 2)^0(x - \omega)^3(x - \overline{\omega})^3
\]

\[
\tag{4} g(x) = (x - 1)^3(x + 2)^2(x - 2)^3(x - \omega)^4(x - \overline{\omega})^4.
\]

Thus it follows by the GCD-formula that

\[
\gcd(f, g) = (x - 1)^{\min(5,3)}(x + 2)^{\min(2,2)}(x - 2)^{\min(0,3)}(x - \omega)^{\min(3,4)}(x - \overline{\omega})^{\min(3,4)}
= (x - 1)^3(x + 2)^2(x - \omega)^3(x - \overline{\omega})^3 = (x - 1)^3(x + 2)^2(x^2 + x + 1)^3.
\]
7. By Theorem 3.1 we know that the equation $z^5 = 1$ has 5 distinct solutions $z_0, \ldots, z_4$ in $\mathbb{C}$ which are given by the formula

$$z_k = \cos \left( \frac{2\pi k}{5} \right) + i \sin \left( \frac{2\pi k}{5} \right), \quad \text{for } k = 0, \ldots, 4.$$ 

Thus, $f(x) = x^5 - 1$ has the 5 distinct roots $z_0, \ldots, z_4$. Since $f(z)$ is monic of degree 5, it follows that

$$x^5 - 1 = (x - z_0)(x - z_1)(x - z_2)(x - z_3)(x - z_4) = (x - 1)(x - \cos(\frac{2\pi}{5}) - i \sin(\frac{2\pi}{5}))(x - \cos(\frac{4\pi}{5}) - i \sin(\frac{4\pi}{5}))(x - \cos(\frac{6\pi}{5}) - i \sin(\frac{6\pi}{5}))(x - \cos(\frac{8\pi}{5}) - i \sin(\frac{8\pi}{5})).$$