Math 211

Assignment 3 - Solutions

1. Let $x$ denote the number of children and $y$ the number of adults that attended. Then we have $375x + 900y = 16500$. The Euclidean algorithm yields

\[
\begin{align*}
900 &= 2 \cdot 375 + 150 \\
375 &= 2 \cdot 150 + 75 \\
150 &= 2 \cdot 75 + 0
\end{align*}
\]

so $\gcd(900, 375) = 75$. Since $75 | 16500$, we see that integer solutions exist. By back-substitution we get $16500 = 375 \cdot 44 + 75$, so $375 = 375 - 2(900 - 2 \cdot 375) = 5 \cdot 375 - 2 \cdot 900$, so $x_0 = 5$ and $y_0 = -2$ satisfy $375x_0 + 900y_0 = 75$. Thus, the general solution is:

\[
\begin{align*}
x &= \frac{16500}{75}(5) + \frac{900}{75}t = 1100 + 12t \\
y &= \frac{16500}{75}(-2) - \frac{75}{75}t = -440 - 5t
\end{align*}
\]

where $t \in \mathbb{Z}$.

We now analyze the constraints. Since $x \geq 0$ we have $1100 + 12t \geq 0$, or $t \geq -\frac{1100}{12} = -91.6667$, so $t \geq -91$. Moreover, since also $y > x$ we have $-440 - 5t > 1100 + 12t$ or $t < -\frac{1540}{17} = -90.58823529$, so $t \leq -91$. Thus, $t = -91$ and hence $x = 1100 + 12(-91) = 8$. (Similarly, $y = 15$.) Thus, there were 8 children at the theatre.

2. Let $x$ denote the cent portion on the original cheque, and $y$ the dollar portion. Then the total amount (in cents) on the original cheque is $100y + x$, and the amount paid out was $100x + y$. We thus obtain $100x + y - 350 = 2(100y + x)$, or equivalently,

\[
(1) \quad 98x - 199y = 350.
\]

Note that $x$ and $y$ also satisfy the constraints

\[
(2) \quad 0 \leq x \leq 99, \quad 0 \leq y \leq 99.
\]

The Euclidean algorithm gives

\[
199 = 2 \cdot 98 + 3, \quad 98 = 32 \cdot 3 + 2, \quad 3 = 1 \cdot 2 + 1.
\]

Thus, $\gcd(199, 98) = 1$ and $1 = 3 - 2 = 3 - (98 - 32 \cdot 3) = 33 \cdot 3 - 98 = 33(199 - 2 \cdot 98) - 98 = (-67) \cdot 98 + 33 \cdot 199 = (-67) \cdot 98 + (-33) \cdot (-199)$. Therefore, the general solution of (1) is

\[
\begin{align*}
x &= 350(-67) + (-199)t = -23450 - 199t \\
y &= 350(-33) - 98t = -11550 - 98t
\end{align*}
\]

where $t \in \mathbb{Z}$.

We now analyze the constraints (2):

\[
x \geq 0 \iff t \leq -\frac{350}{199} \cdot 67 = -117.84 \iff t \leq -118;
\]

\[
x \leq 99 \iff t \geq -\frac{350}{199} \cdot (-99) = -118.33 \iff t \geq -118.
\]

Thus, $t = -118$ and hence $x = 350(-67) - 199(-118) = 32$, and $y = 350(-33) - 98(-118) = 14$. Therefore, the cheque was worth $14.32.

3. (a) Write the equation as $4x + 6y = 20 + 9z$. Since $\gcd(4, 6) = 2$, we see by the GCD-criterion that this has a solution if and only if $2|20 + 9z$, i.e. if and only if $20 + 9z = 2w$, for some $w \in \mathbb{Z}$. Now since $\gcd(2, 9) = 1$ and $2(-4) - 9(-1) = 1$, the general solution of the auxiliary equation $2w - 9z = 20$ is given by

\[
w = \frac{20}{1}(-4) + \frac{9}{1}t = -80 - 9t, \quad z = \frac{20}{1}(-1) - \frac{2}{1}t = -20 - 2t, \quad \text{for } t \in \mathbb{Z}.
\]
4. Let $x$

Substituting this value for $z$ in the original equation yields $4x + 6y = 20 + 9(-20 - 2t) = -160 - 18t$. Now since $\gcd(4, 6) = 2$ and $4(-1) + 6(1) = 2$, we see that the general solution of $4x + 6y = -160 - 18t$ is given by

$$x = \frac{-160 - 18t}{2}(-1) + \frac{5}{2}s = 80 + 9t + 3s, \quad y = \frac{-160 - 18t}{2}(1) - \frac{4}{2}s = -80 - 9t - 2s, \quad \text{where } s, t \in \mathbb{Z}.$$ 

Thus, the general solution of the original equation is

$$x = 80 + 9t + 3s, \quad y = -80 - 9t - 2s, \quad z = -20 - 2t, \quad \text{where } s, t \in \mathbb{Z}.$$ 

Check: $4(80 + 9t + 3s) + 6(-80 - 9t - 2s) - 9(-20 - 2t) = 320 + 36t + 12s - 480 - 54t - 12s + 180 + 18t = 20.$ 

(b) This equation has no integral solutions. Indeed, since 3 divides 3, 9, and 12, we see (by property (D3)) that $3|3x + 9y + 12z$ for all integers $x, y, z$. However, 3 does not divide 20, so there are no integers $x, y, z$ such that $3x + 9y + 12z = 20$.

4. Let $x$ denote the value of a six-pence coin, $y$ the value of a ten-centime coin and $z$ the value of a drachma coin (all in cents). Then we have the equation

$$35x + 55y + 77z = 586,$$

with constraints $x \geq 0, y \geq 0$ and $z \geq 0$.

To solve this equation, write it as:

$$35x + 55y = 586 - 77z.$$ 

Since $\gcd(35, 55) = 5$, this has a solution if and only if $5|(586 - 77z)$ or, equivalently, if

$$77z + 5w = 586, \quad \text{for some } w \in \mathbb{Z}.$$ 

The Euclidean algorithm gives $77 = 15 \cdot 5 + 2, 5 = 2 \cdot 2 + 1$; thus, $\gcd(77, 5) = 1$ and, moreover, $1 = 5 - 2 \cdot 2 = 5 - 2(77 - 15 \cdot 5) = (-2) \cdot 77 + 31 \cdot 5$. Thus, the general solution of (5) is:

$$z = 586(-2) + 5t, \quad w = 586 \cdot 31 - 77t, \quad \text{where } t \in \mathbb{Z}.$$ 

Substituting for $z$ is (4) yields $35x + 55y = 155 \cdot 586 - 77 \cdot 5t$ or

$$7x + 11y = 31 \cdot 586 - 77t.$$ 

Since $\gcd(7, 11) = 1$ and $7(-3) + 11(2) = 1$, the general solution of (6) is

$$x = (31 \cdot 586 - 77t)(-3) + 11s, \quad y = (31 \cdot 586 - 77t)(2) - 7s,$$

and hence the general solution solution of (3) is

$$\begin{align*}
x &= -3 \cdot 31 \cdot 586 + 3 \cdot 77t + 11s, \\
y &= 2 \cdot 31 \cdot 586 - 2 \cdot 77t - 7s, \\
\end{align*}$$ 

where $s, t \in \mathbb{Z}$.

(Other expressions are also possible. Note that $3 \cdot 31 \cdot 586 = 54498, 3 \cdot 77 = 231, 2 \cdot 586 = 1172, 2 \cdot 31 \cdot 586 = 36332$ and $2 \cdot 77 = 154$, but we don’t need this.)

We now analyse the above constraints. We have:

$$\begin{align*}
z \geq 0 & \iff t \geq (2 \cdot 586)/5 = 234.6 & \iff t \geq 235 \text{ (since } t \in \mathbb{Z}). \\
y \geq 0 & \iff s \leq -22t + (2 \cdot 31 \cdot 586)/7 = -22t + 5190 \frac{2}{7} & \iff s \leq -22t + 5190, \\
x \geq 0 & \iff s \geq -21t + (3 \cdot 31 \cdot 586)/11 = -21t + 4954 \frac{4}{11} & \iff s \geq -21t + 4955. \\
\end{align*}$$ 

3–2
We thus have the inequalities

\[ -22t + 5190 \geq s \geq -21t + 4955 \quad \text{and} \quad t \geq 235. \]

From these inequalities we obtain \(-22t + 5190 \geq -21t + 4955\) or \(t \leq 5190 - 4955 = 235\), which, together the above inequality on \(t\) yields \(t = 235\). Substituting this into the inequalities (7) gives

\[ s = -22(235) + 5190 \geq -21(235) + 4955 = 20, \]

and hence \(s = 20\). Substituting \(s = 20, t = 235\) into the above formulae gives \(x = -93(586) + 3 \cdot 77 \cdot 235 + 11 \cdot 20 = 7, y = 62(586) - 154 \cdot 235 - 7 \cdot 20 = 2, z = (-2)586 + 5(235) = 3,\) and so the value of six-pence coin was \(7\mathfrak{f}\), that of the ten-centime coin was \(2\mathfrak{f}\), and that of the drachma was \(3\mathfrak{f}\).

**Alternate method** (for analyzing constraints): The above inequalities (7) describe the interior region of a triangle in the \((t, s)\)-plane with vertices \((\frac{18168}{1172}, 0) = (235, 0), (\frac{1722}{1172}, \frac{1753}{1172}) = (234.4, 33.49), (\frac{1172}{1172}, \frac{1753}{1172}) = (234.4, 31.96).\) Thus, any integral point in this region must have \(t = 235\). The line \(t = 235\) meets the triangle at \((235, \frac{1172}{1172}) = (235, 19.4)\) and at \((235, \frac{1753}{1172}) = (235, 20.3)\). Thus we see that \((t, s) = (235, 20)\) is the only point in the triangle and so we obtain the solution \(x = 7, y = 2\) and \(z = 3\) as before.

[2]

5. Rather than writing out all the numbers \(< 120\), it is enough to write down \(2\) and the odd numbers \(< 120\), and cross out successively multiples of \(3, 5, 7\). (Note that \(11 > \sqrt{120}\), so we do not need to cross out any other multiples.) We then obtain the following table (Sieve of Eratosthenes):

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</table>

Thus, the primes less than \(120\) are:

\(2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113\)

and the pairs of twin primes in that range are:

\((3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73), (101, 103), (107, 109)\).

[1]

6. (a) \(1728 = 12 \cdot 144 = 12^3 = 2^63^3;\)

(b) \(684600 = 200 \cdot 3423 = (8 \cdot 5^2)(3 \cdot 7 \cdot 163) = 2^33^3 \cdot 5^2 \cdot 7 \cdot 163.\)

[2]

7. Let \(P(n)\) be the statement: \(1^2 + 2^2 + \ldots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.\) This statement is clearly true for \(n = 1\) because \(\frac{1}{3}1^3 + \frac{1}{2}1^2 + \frac{1}{6}1 = 1 = 1^2.\) Thus, assume that \(P(n)\) is true for \(n = k, \) i.e. assume that \(1^2 + 2^2 + \ldots + k^2 = \frac{1}{3}k^3 + \frac{3}{2}k^2 + \frac{13}{6}k\) (induction hypothesis). To show that \(P(n)\) is also true for \(n = k + 1,\) note that the induction hypothesis implies that

\[ 1^2 + 2^2 + \ldots + k^2 + (k + 1)^2 = (1^2 + 2^2 + \ldots + k^2) + (k + 1)^2 = \left( \frac{1}{3}k^3 + \frac{3}{2}k^2 + \frac{13}{6}k \right) + \left( k + \frac{1}{3} \right) \]

\[ = \frac{1}{3}k^3 + \frac{3}{2}k^2 + \frac{13}{6}k + \frac{1}{3} = \frac{1}{3}(k + 1)^3 + \frac{1}{2}(k + 1)^2 + \frac{1}{6}(k + 1);\]

here we've used in the last step the identities \((k + 1)^3 = k^3 + 3k^2 + 3k + 1\) and \((k + 1)^2 = k^2 + 2k + 1.\) This, therefore, shows that \(P(k + 1)\) is true, and so, by the principle of mathematical induction, we have proved that \(P(n)\) is true for all \(n \geq 1.\)