Math 211

Assignment 4 - Solutions

[3] 1. Let $x = \text{Granny’s age}$, $y = \text{Mrs. Jones’ age}$. Then we have

\[ 1817 = x^2 - y^2 = (x + y)(x - y). \]

Since $1817 = 23 \cdot 79$ is the prime factorization of 1817, we see that its only positive divisors are 1, 23, 79 and 1817, and so $1\cdot 1817 = 23 \cdot 79 = 79 \cdot 23 = 1817 \cdot 1$ are the only factorizations of 1817 into two (positive) factors. Moreover, since $x + y > x - y$ (because $y > 0$), we see that

\[ x + y = 79, \quad x - y = 23 \]

is the only possibility because the solution $x - y = 1$ is impossible as Granny is Mrs. Jones’ mother. This gives $x = 51$, $y = 28$, so Mrs. Jones is 28 years old.

[2] 2. Suppose $\sqrt{40}$ were rational, so $\sqrt{40} = \frac{m}{n}$ with $m, n \in \mathbb{Z}$. Then $40 = \frac{m^2}{n^2}$, or $40n^2 = m^2$. Thus $3\text{exp}(m) = \text{exp}(40n^2) = \text{exp}(40) + \text{exp}(n^3) = 1 + 3\text{exp}(n)$. We thus obtain that $1 + 3\text{exp}(n) = 3\text{exp}(m)$ which is impossible since 3 does not divide 1. Thus, no such integers $m, n$ exist and hence $\sqrt{40}$ is irrational.

Note: We could not have obtained a contradiction by using $\text{exp}(2)$ or $\text{exp}(3)$ in place of $\text{exp}(5)$. Similarly, an odd/even argument does not work here; instead, we need to discuss divisibility by 3.

[3] 3. (a) By the GCD-formula we have

\[ \text{gcd}(2^53^711^2, 3^25^37^413^2) = 2^{\min(5,0)}3^{\min(7,2)}5^{\min(0,3)}7^{\min(0,4)}11^{\min(3,0)}13^{\min(0,2)}17^{\min(2,0)} = 3^2. \]

(b) By the first part of Theorem 9 (GCD-formula), every divisor of $n = 3^25^37^413^2$ has the (unique) form

\[ \pm 3^a5^b7^c13^d, \quad \text{where } 0 \leq a \leq 2, \ 0 \leq b \leq 3, \ 0 \leq c \leq 4, \ 0 \leq d \leq 2. \]

Thus, there are $(2 + 1)(3 + 1)(4 + 1)(2 + 1) = 180$ positive divisors and 360 divisors in total. The first few are: $\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 13, \pm 15, \pm 21, \ldots$; these correspond to the values $(a, b, c, d) = (0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (2, 0, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 1, 0)$, respectively.

(c) The prime decomposition of $mn$ is $mn = 2^55^37^411^317^213^2$.

[5] 4. (a) Since $2^{12} = (2^6)^2 = 64^2 \equiv 1 \equiv 1 \mod(63)$, we have $\text{rem}(2^{12}, 63) = 1$.

(b) Since $9 + 8 = 17$, we have $9 \equiv -8 \mod(17)$, and so $9^5 \equiv -8^5 \mod(17)$. But $(-8)^5 = -(8^5)$, so $9^5 \equiv -8^5 \mod(17)$ or $9^5 + 8^5 \equiv 0 \mod(17)$. Thus $\text{rem}(8^5 + 9^5, 17) = 0$.

(c) Since $3^2 = 9 \equiv -1 \mod(10)$, we have $3^8 = (3^2)^4 \equiv (-1)^4 \equiv 1 \mod(10)$. Similarly, $4^8 = 16^4 \equiv (-4)^4 = 16^2 \equiv (-4)^2 \equiv 16 \equiv -4 \mod(10)$. Thus, $3^8 - 4^8 \equiv 1 - (-4) \equiv 5 \mod(10)$ and so $\text{rem}(3^8 - 4^8, 10) = 5$.

(d) We have $24 \cdot 25 + 27 \cdot 29 \equiv (-2)(-1) + (1)(3) \equiv 5 \mod(26)$, so $\text{rem}(24 \cdot 25 + 27 \cdot 29, 26) = 5$.

(e) Here $103 \cdot 65 - 329 \cdot 663 \equiv 4(-1) - (-1)(3) \equiv -1 \equiv 32 \mod(33)$, so $\text{rem}(103 \cdot 65 - 329 \cdot 663, 33) = 32$.

[2] 5. We have $234785346 \equiv (2 + 3 + 4) + 7 + 8 + (5 + 4) + (3 + 6) \equiv 15 \equiv 6 \mod(9)$ and $5683592187 \equiv 5 + 6 + 3 + (8+1)+5+(9) + (2+7)+8 \equiv 18 \equiv 0 \mod(9)$. Thus $234785346 \cdot 5683592187 \equiv 6 \cdot 0 \equiv 0 \mod(9)$.

On the other hand, $1334424157147691702 \equiv (1 + 3 + 3 + 2) + (4 + 4 + 1) + (4 + 5) + (1 + 7 + 1) + 4 + 7 + 6 + (9) + 1 + (7 + 0 + 2) \equiv 17 \equiv 8 \mod(9)$, and so $234785346 \cdot 5683592187 \neq 1334424157147691702$. In fact, $234785346 \cdot 5683592187 = 1334424157147691702$.