Assignment 7 - Solutions

1. (a) $(2 + 3i)(3 - 2i) = 2(3 - 2i) + 3i(3 - 2i) = 6 - 4i + 9i - 6i^2 = 6 + 5i - 6(-1) = 12 + 5i.$

[Alternately: by the formula we have $(2 + 3i)(3 - 2i) = (2(3) - 3(-2)) + i(2(-2) + 3(3)) = 12 + 5i.$]

(b), (c) By multiplying top and bottom by the complex conjugate of the denominator, we get:

\[ \frac{1+i(2-3i)}{3+i} = \frac{(1+i)(2-3i)}{3^2+1^2} = \frac{(5-i)(3-i)}{10} = \frac{7}{5} - \frac{4i}{5}. \]

\[ \frac{2(-8+6i)^2}{(4-2\sqrt{5})(2-4\sqrt{5})} = \frac{2(-8+6i)(4+2\sqrt{5})(2+4\sqrt{5})}{36(1-12)} = \frac{2(-28-96)(-32+4i(20\sqrt{5}))}{36\sqrt{5}} \]
\[ = \frac{2}{2\sqrt{5}} \left[ (7 - 24i)(-8 + i5\sqrt{5}) \right] = \frac{2}{2\sqrt{5}} \left[ (-24 \cdot 8 - 7 \cdot 5\sqrt{5}) + i(24 \cdot 5\sqrt{5} - 8 \cdot 7) \right] \]
\[ = -\left( \frac{128}{63} + \frac{10}{27}\sqrt{5} \right) + i \left( \frac{80}{63}\sqrt{5} - \frac{16}{27} \right) \] (Alternately: $= -\left( \frac{384+70\sqrt{5}}{189} \right) + \frac{-112+240\sqrt{5}}{189}i.$)

2. (a) $\alpha = \sqrt{2}(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$, $\beta = 2(\cos(\frac{\pi}{8}) + i\sin(\frac{\pi}{8})).$

(b) Using part (a) and the multiplication formula we obtain that $\alpha\beta = 2\sqrt{2}(\cos(\frac{\pi}{4} + \frac{\pi}{8}) + i\sin(\frac{\pi}{4} + \frac{\pi}{8})) = 2\sqrt{2}(\cos(\frac{7\pi}{16}) + i\sin(\frac{7\pi}{16})).$ Similarly, $\beta/\alpha = (2/\sqrt{2})(\cos(\frac{\pi}{8} - \frac{\pi}{8}) + i\sin(\frac{\pi}{8} - \frac{\pi}{8})) = \sqrt{2}(\cos(\frac{\pi}{16}) + i\sin(\frac{\pi}{16})).$

(c) Since $1 - i$ is in the fourth quadrant, we obtain $\arg(1 - i) = 2\pi + \arctan(-\frac{1}{1}) = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}.$ Similarly, since $-1 + i\sqrt{3}$ lies in the second quadrant, we have $\arg(-1 + i\sqrt{3}) = \arctan(\frac{\sqrt{3}}{1}) + \pi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}.$

Note: Do not use approximations in your calculations unless this is necessary.

3. (a) Put $z_1 = x + iy$, $z_2 = a + bi$. Then

\[ (x^2 + y^2)(a^2 + b^2) = |z_1|^2|z_2|^2 = z_1\overline{z}_2 \overline{z}_2 = (z_1z_2)\overline{z}_2 = |z_1z_2|^2 = \text{Re}(z_1z_2)^2 + \text{Im}(z_1z_2)^2. \]

From the formula in class we know that $\text{Re}(z_1z_2) = xa - yb$ and $\text{Im}(z_1z_2) = xb + ya$, so the identity follows.

(b) By hypothesis, $n = x^2 + y^2$, $m = a^2 + b^2$ with $x, y, a, b \in \mathbb{Z}$. From the formula in (a) we obtain that $nm = (xa - yb)^2 + (xb + ya)^2$ is the sum of two integer squares since the sum and product of integers is again an integer.

4. See the MAPLE solution on the course Web site (www.mast.queensu.ca/~math211).