Problem 1:

(a) For any \(a, b \in \mathbb{Z}\),
\[
\alpha(a + b) = [a + b] = [a] + [b] = \alpha(a) + \alpha(b).
\]
Thus \(\alpha\) is a homomorphism. The kernel of \(\alpha\) is
\[
Ker(\alpha) = \{a \in \mathbb{Z} : \alpha(a) = [0]\}
= \{a \in \mathbb{Z} : [a] = [0]\}
= \{a \in \mathbb{Z} : a \equiv 0 \text{ (modulo } n)\}
= \{a \in \mathbb{Z} : a = nq, q \in \mathbb{Z}\}
= n\mathbb{Z}.
\]
\(\alpha\) is onto since for any \([a] \in \mathbb{Z}_n\), \([a] = \alpha(a)\) as \(a\) is itself an integer (varying from 0 to \(n - 1\)). However, \(\alpha\) is not 1-1, since for example \(\alpha(0) = \alpha(n) = [0]\).

(b) For any \(a, b \in G\), \(\alpha(ab) = (ab)^{-1} = b^{-1}a^{-1}\). But \(\alpha(a)\alpha(b) = a^{-1}b^{-1}\). Since in general, \(b^{-1}a^{-1} \neq a^{-1}b^{-1}\) (unless if \(G\) is abelian), then \(\alpha\) is not a homomorphism.
\(\alpha\) is onto since for any \(b \in G\), \(b^{-1} \in G\) and \(\alpha(b^{-1}) = (b^{-1})^{-1} = b\). Thus for any \(b \in G\), there exists \(a = b^{-1} \in G\) such that \(b = \alpha(a)\). Also, \(\alpha\) is 1-1 since for any \(a_1, a_2 \in G\),
\[
\alpha(a_1) = \alpha(a_2) \implies a_1^{-1} = a_2^{-1} \implies (a_1^{-1})^{-1} = (a_2^{-1})^{-1} \implies a_1 = a_2.
\]

(c) Since \(G\) is abelian, we directly obtain (from the discussion in (b) above) that \(\alpha\) is a homomorphism. Actually since \(\alpha\) is 1-1 and onto (as shown in (b)), it is an isomorphism. Its kernel is is given by
\[
Ker(\alpha) = \{g \in G : \alpha(g) = e\}
= \{g \in G : g^{-1} = e\}
= \{e\},
\]
which is expected since \(\alpha\) is 1-1.
(d) For all \( x, y \in \mathbb{R} \setminus \{0\} \),

\[
\alpha(xy) = \begin{cases} 
1 & \text{if } xy > 0 \iff (x > 0, y > 0) \text{ or } (x < 0, y < 0) \\
-1 & \text{if } xy < 0 \iff (x < 0, y > 0) \text{ or } (x > 0, y < 0)
\end{cases}
\]

It can be clearly checked that \( \alpha(xy) = \alpha(x)\alpha(y) \) for all above cases; thus \( \alpha \) is a homomorphism and its kernel is given by

\[
\text{Ker}(\alpha) = \{ x \in \mathbb{R} \setminus \{0\} : \alpha(x) = 1 \} = \{ x \in \mathbb{R} \setminus \{0\} : x > 0 \} = \mathbb{R}^+
\]

which is the set of positive real numbers.

\( \alpha \) is onto since \( 1 = \alpha(1) \) and \( -1 = \alpha(-1) \), but \( \alpha \) is not 1-1 since for example \( \alpha(1) = 1 = \alpha(2) \).

(e) For any \( a, b \in G \),

\[
\alpha(ab) = (ab)^n = a^n b^n = \alpha(a)\alpha(b)
\]

where the second equality holds since \( G \) is abelian. Thus \( \alpha \) is a homomorphism and its kernel is given by

\[
\text{Ker}(\alpha) = \{ g \in G : \alpha(g) = e \} = \{ g \in G : g^n = e \} = \{ g \in G : |g| \text{ divides } n \}.
\]

\( \alpha \) is not onto since for example let \( G = \mathbb{R} \setminus \{0\} \) (under the multiplication operation) and let \( n = 2 \); then for any negative real \( y \in G \), there does not exist an \( x \in G \) such that \( y = x^2 \).

Also \( \alpha \) is not 1-1: let \( a \neq e \) be in \( G \) such that \( |a| \text{ divides } n \); then

\[
\alpha(a) = a^n = e = \alpha(e);
\]

(e.g., let \( n = 6, G = \mathbb{Z}_6 \) with \( e = [0] \) and \( a = [2] \)).

**Problem 2:**

Let us first show that \( G \) is a group under matrix multiplication.

- For any \( a, b \in G \), we can write

\[
a = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix},
\]
where \( n, m \in \mathbb{Z} \). Then
\[
\begin{pmatrix}
1 & n \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & m \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & n + m \\
0 & 1
\end{pmatrix}
\]
is in \( G \) since \( n + m \in \mathbb{Z} \). Thus \( G \) is closed.

- Matrix multiplication is associative.
- The unity element is the identity matrix
\[
I = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
which is in \( G \).
- For any
\[
\begin{pmatrix}
1 & n \\
0 & 1
\end{pmatrix}
\in G,
\]
we have
\[
\begin{pmatrix}
1 & n \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -n \\
0 & 1
\end{pmatrix}
= I = \begin{pmatrix}
1 & -n \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & n \\
0 & 1
\end{pmatrix}.
\]
Thus
\[
\begin{pmatrix}
1 & n \\
0 & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
1 & -n \\
0 & 1
\end{pmatrix} \in G.
\]
Therefore \( G \) is a group. Now define the mapping \( \alpha : \mathbb{Z} \rightarrow G \) by
\[
\alpha(n) = \begin{pmatrix}
1 & n \\
0 & 1
\end{pmatrix}
\]
for all \( n \in \mathbb{Z} \). This mapping is clearly well-defined, 1-1 and onto (check it); also it is a homomorphism since given \( m, n \in \mathbb{Z} \), we have
\[
\alpha(m + n) = \begin{pmatrix}
1 & m + n \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & m \\
0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 & n \\
0 & 1
\end{pmatrix} = \alpha(m) \cdot \alpha(n).
\]
Thus \( \alpha \) is an isomorphism and \( \mathbb{Z} \cong G \). \( \square \)
Problem 3:
First verify that $G$ is a subgroup of $GL_2(\mathbb{Z})$ by checking that it is closed (it is enough to check closure since $G$ is finite); this can be directly verified by constructing the Cayley table of $G$. Also from the Cayley table, we directly can note that if we set
\[ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]
Then
\[ A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I = A^0. \]
Thus $G = \langle A \rangle = \{ A^0 = I, A, A^2, A^3 \}$. Noting also that
\[ H \uparrow \triangleq \{ 1, -1, i, -i \} = \{ i^0 = 1, i, i^2, i^3 \} = \langle i \rangle, \]
we can directly construct an isomorphism (verify it) $\alpha : G \rightarrow H$ defined by
\[ \alpha(A^k) = i^k, \quad k = 0, 1, 2, 3. \]
Thus $G \cong H$. \qed

Problem 4:
Let $G = \langle a \rangle$ with $|G| = |a| = \infty$. Then define $\alpha : \mathbb{Z} \rightarrow G$ by
\[ \alpha(k) = a^k \]
for any $k \in \mathbb{Z}$.

Note first that $\alpha$ is a well-defined mapping since every $k$ in $\mathbb{Z}$ has an image in $G$ and for any $k, m \in \mathbb{Z}$,
\[ k = m \implies a^k = a^m \implies \alpha(k) = \alpha(m). \]
Now for any $k, m \in \mathbb{Z}$,
\[ \alpha(k + m) = a^{k+m} = a^k a^m = \alpha(k) \alpha(m), \]
and hence $\alpha$ is a homomorphism. Also, $\alpha$ is clearly onto since for any $a^k \in G$, $a^k = \alpha(k)$ where $k \in \mathbb{Z}$. Finally, for any $k, m \in \mathbb{Z}$
\[ \alpha(k) = \alpha(m) \implies a^k = a^m \implies a^{k-m} = e. \]
But $a^{k-m} = e$ iff $k = m$ since $|a| = \infty$; hence $\alpha$ is 1-1. Therefore $G \cong \mathbb{Z}$. \qed
Problem 5:

(a) If $\alpha$ is an isomorphism, then for all $a, b \in G$,

\[
a^{-1}b^{-1} = \alpha(a)\alpha(b) = \alpha(ab) \quad \text{(as $\alpha$ is a homomorphism)}
= (ab)^{-1}
= b^{-1}a^{-1}.
\]

Thus $ba = ab$ for all $a, b \in G$ and $G$ is abelian.

Conversely, if $G$ is abelian, then for all $a, b \in G$,

\[
\alpha(ab) = (ab)^{-1}
= b^{-1}a^{-1}
= a^{-1}b^{-1} \quad \text{(since $G$ is abelian)}
= \alpha(a)\alpha(b)
\]

and hence $\alpha$ is a homomorphism.

Let us show that $\alpha$ is 1-1: for $a, b \in G$, $\alpha(a) = \alpha(b)$ implies that $a^{-1} = b^{-1}$. Thus $a = b$ and $\alpha$ is 1-1.

Let us finally show that $\alpha$ is onto. For any $b \in G$, $b = (b^{-1})^{-1} = \alpha(b^{-1})$ where $b^{-1} \in G$ as $G$ is a group. Thus $\alpha$ is onto.

We have thus shown that if $G$ is abelian, then $\alpha$ is an isomorphism. \qed

(b) Let $\alpha : G \to G_1$ and $\beta : H \to H_1$ be isomorphisms. Define $\gamma : G \times H \to G_1 \times H_1$ by

\[
\gamma(g, h) = (\alpha(g), \beta(h))
\]

for all $(g, h) \in G \times H$.

Recall that $G \times H$ and $G_1 \times H_1$ are each a group under the component-wise operation.

The mapping $\gamma$ is onto, since for any $(g_1, h_1)$ in $G_1 \times H_1$, there exist $g \in G$ and $h \in H$ such that $g_1 = \alpha(g)$ and $h_1 = \beta(h)$, since $\alpha$ and $\beta$ are each onto; thus there exists $(g, h)$ in $G \times H$ such that $(g_1, h_1) = (\alpha(g), \beta(h)) = \gamma(g, h)$.

Also, $\gamma$ is 1-1, since for $(g, h)$ and $(g', h')$ in $G \times H$ such that $\gamma(g, h) = \gamma(g', h')$, we have directly (by definition of $\gamma$) that $\alpha(g) = \alpha(g')$ and $\beta(h) = \beta(h')$. Now since both $\alpha$ and $\beta$ are 1-1, we obtain that $g = g'$ and $h = h'$; thus $(g, h) = (g', h')$ and $\gamma$ is onto.
Finally, $\gamma$ is a homomorphism since

\[
\gamma((g,h)(g',h')) = \gamma((gg', hh')) \quad \text{(using the component-wise operation of $G \times H$)}
\]

\[
= (\alpha(gg'), \beta(hh'))
\]

\[
= (\alpha(g)\alpha(g'), \beta(h)\beta(h')) \quad \text{(as $\alpha$ and $\beta$ are each a homomorphism)}
\]

\[
= (\alpha(g), \beta(h))(\alpha(g'), \beta(h')) \quad \text{(using the component-wise operation of $G_1 \times H_1$)}
\]

\[
= \gamma((g,h))\gamma((g',h'))
\]

and hence $\gamma$ is a homomorphism.

This $\gamma$ is an isomorphism. \qed