SOLUTIONS
Given sets $A = \{x, y\}$ and $B = \{0, 1\}$, determine the set $P(A) \times P(B)$, where $P(\cdot)$ denotes the power set.

**Solution:**

Noting that

\[ P(A) = \{\emptyset, \{x\}, \{y\}, A\} \]

and

\[ P(B) = \{\emptyset, \{0\}, \{1\}, B\}, \]

we have

\[ P(A) \times P(B) = \{(\emptyset, \emptyset), (\emptyset, \{0\}), (\emptyset, \{1\}), (\emptyset, B), (\{x\}, \emptyset), (\{x\}, \{0\}), (\{x\}, \{1\}), \]
\[ (\{y\}, \emptyset), (\{y\}, \{0\}), (\{y\}, \{1\}), (\{y\}, B), (A, \emptyset), (A, \{0\}), (A, \{1\}), (B, B)\}. \]
2. Given functions $\alpha : A \to B$ and $\beta : B \to C$ such that $\beta \alpha$ is onto and $\beta$ is one-to-one, show that $\alpha$ is onto.

Solution:

Let $b \in B$, then $\beta(b) \in C$. Since $\beta \alpha$ is onto, then $\exists a \in A$ such that $\beta(b) = \beta \alpha(a)$. Thus $\beta(b) = \beta(\alpha(a))$ by the definition of composition. The latter implies that $b = \alpha(a)$ since $\beta$ is $1-1$. We have hence shown that for any $b \in B$, $\exists a \in A$ such that $b = \alpha(a)$; so $\alpha$ is onto. \(\square\)
3. State the definition of an equivalence relation $R$ on a set $A$ and show that for $a, b \in A$, $[a] = [b]$ iff $aRb$. [6]

**Solution:**

A relation $R$ from set $A$ to set $B$ is a subset of $A \times B$: $R \subseteq A \times B$. If the sets $A$ and $B$ are equal ($A = B$), then $R \subseteq A \times A$ is called a relation on set $A$.

Also, a relation $R$ on a set $A$ is said to be an equivalence relation on the set $A$ if it satisfies the following three conditions.

(i) $R$ is reflexive: For every $a \in A$, $aRa$.

(ii) $R$ is symmetric: If there exist $a, b \in A$ such that $aRb$, then $bRa$.

(iii) $R$ is transitive: If there exist $a, b, c \in A$ such that $aRb$ and $bRc$, then $aRc$.

We next prove that for $a, b \in A$, $[a] = [b]$ iff $aRb$.

In order to show the forward direction, we assume that $aRb$; we aim to show $[a] = [b]$ by double inclusion. Take an arbitrary element $c \in [b]$. Since $c \in [b]$ we have $bRc$. As $aRb$, we know $aRc$ by transitivity. Therefore $c \in [a]$. We have shown $[b] \subseteq [a]$. The reverse containment $[a] \subseteq [b]$ holds by symmetry. Thus, $[a] = [b]$.

For the opposite direction, assume that $[a] = [b]$. Since $bRb$, we have $b \in [b] = [a]$ and hence $aRb$. Therefore $[a] = [b]$ implies $aRb$. $\square$
4. For a function \( f : A \rightarrow A \), let \( f^n = f \cdot f \cdots f : A \rightarrow A \) be the composition of \( f \) with itself \( n \) times. Use induction to prove that, if \( f \) is one-to-one, then \( f^n \) is one-to-one for all integers \( n \geq 1 \).

**Solution:**

We use induction on \( n \geq 1 \) to show the result.

- **Base case:** For \( n = 1 \), \( f^1 = f \) is one-to-one (by assumption); this proves the base case.

- **Inductive case:** For arbitrary integer \( k \geq 1 \), assume that \( f^k \) is one-to-one. Let us show that \( f^{k+1} \) is one-to-one. For \( a_1, a_2 \in A \), we have

\[
\begin{align*}
f^{k+1}(a_1) = f^{k+1}(a_2) & \implies f(f^k(a_1)) = f(f^k(a_2)) \quad \text{(by composition definition)} \\
& \implies f(a_1) = f(a_2) \quad \text{(since \( f^k \) is one-to-one)} \\
& \implies a_1 = a_2 \quad \text{(since \( f \) is one-to-one)}.
\end{align*}
\]

Thus \( f^{k+1} \) is one-to-one.

Hence, by induction, \( f^n \) is one-to-one for all integers \( n \geq 1 \).
5. Given two relatively prime integers $m$ and $n$ and an integer $k$ such that $n | (mk)$, show that $n | k$. \[6\]

Solution:

Since $gcd(m, n) = 1$, then there exist integers $x$ and $y$ such that

$$xm + yn = 1$$

by Bezout’s identity. Thus

$$xmk + ynk = k.$$ 

Since $mk = nq$ for some integer $q$ (as $n | (mk)$), then plugging this identity in the above equation yields that

$$x(nq) + ynk = k$$

or equivalently,

$$(xq + yk)n = k.$$ 

Since $xq + yk$ is an integer, we directly obtain that $n | k$. \[\square\]