MTHE 217 - Algebraic Structures with Applications
Fall 2017

Homework # 3

Material: Equivalences Relations and Functions

Readings: Section 3 (Lecture Notes) and Sections 2.3 and 2.2 (Humphreys-Prest text)

1. Define the relation $E$ on the set of reals $\mathbb{R}$ by $x E y$ iff $x - y \in \mathbb{Z}$, where $x, y \in \mathbb{R}$ and $\mathbb{Z}$ denotes the set of integers. Show that $E$ is an equivalence relation.

2. In each case decide whether the relation $R$ is an equivalence relation on the set $A$. Justify your answer. If it is an equivalence, describe the equivalence classes.
   (a) $A = \{-1, 0, 1\}$; $a R b$ iff $a^2 = b^2$.
   (b) $A = P(B)$, the power set of the set $B = \{1, 2, 3\}$; $X R Y$ iff $|X| = |Y|$.
   (c) $A = \mathbb{R} \times \mathbb{R}$; $(x, y) R (x_1, y_2)$ iff $x^2 + y^2 = x_1^2 + y_2^2$.

3. Let $U = \{1, 2, 3\}$ and $A = U \times U$. In each case show that the relation $E$ is an equivalence relation on $A$ and find the quotient set $A_E$.
   (a) $(a, b) E (a_1, b_1)$ iff $ab = a_1 b_1$.
   (b) $(a, b) E (a_1, b_1)$ iff $a - b = a_1 - b_1$.

4. In each case determine whether $\alpha$ is a legitimate function. Justify your answer.
   (a) $\alpha : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ defined by $\alpha(xy) = (x, y)$ for all $xy \in \mathbb{R}$.
   (b) $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ defined by $\alpha(n) = 1$ for all $n \in \mathbb{N}$.
   (c) $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\alpha(x) = \sqrt{x}$ for all $x \in \mathbb{R}$.

5. Let $S$ be the set of the following non-negative rational numbers:
   
   $S = \{m/n \in \mathbb{Q} \mid m, n \text{ non-negative integers, } n \neq 0\},$
   
   and let $\mathbb{R}$ be the set of all real numbers.
(a) Does \( f : S \to \mathbb{R} \) defined by \( f(m/n) = 2^m 3^n \) define a legitimate function from \( S \) to \( \mathbb{R} \)?

(b) If not, how could you modify the definition of \( f \) so as to get a legitimate function?

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**Recommended Practice Problems:** (Do not hand in)

1. Page 115, # 1 (a), (b), (d), (e), (f) and (g). (Humphreys-Prest, 2nd Ed.) Check only the reflexivity, symmetry and transitivity properties.
2. Page 116, # 7. (Humphreys-Prest, 2nd Ed.)
3. Page 116, # 8. (Humphreys-Prest, 2nd Ed.)
4. Page 102, # 1. (Humphreys-Prest, 2nd Ed.)
5. Page 102, # 2. (Humphreys-Prest, 2nd Ed.)
6. Page 102, # 3. (Humphreys-Prest, 2nd Ed.)