Queen’s University
Department of Mathematics and Statistics
MATH 217 - Algebraic Structures with Applications

Final Exam
December 5, 2014
Dr. F. Alajaji

Student Number: ________________

Directions

• Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.

• This material is copyrighted and is for the sole use of students registered in MTHE 217 and writing this examination. This material shall not be distributed or disseminated. Failure to abide by these conditions is a breach of copyright and may constitute a breach of academic integrity under the University Senate’s Academic Integrity Policy Statement.

• Total points = 60. Duration = 3 hours.

• Closed Book, Closed Notes, No Calculators Permitted.

• Answers should be put in the space provided.

• Be lucid and neat. Justify all your answers.

• Good Luck!

Marks: Please do not write in the space below.

Problem 1 [8] Problem 5 [8]
Problem 4 [8] Problem 8 [10]
Total: [60]
1. In each case, either prove the statement or disprove it via a counterexample.

(a) ∀ \( m \in \mathbb{Z} \), \( 4m \) and \( 2m + 3 \) are relatively prime. [4]

(b) ∀ \( m \in \mathbb{Z} \), \( 5m + 1 \) and \( 6m + 1 \) are relatively prime. [4]
(Problem 1 - Cont’d)
2. Answer the following questions.

(a) Solve the congruence $4x \equiv 3 \pmod{5}$. [4]

(b) Find the remainder when $11^{717}$ is divided by 7. [4]
(Problem 1 - Cont’d)
3. Let $G$ be a group. For fixed element $a \in G$, the centralizer of $a$ in $G$ is defined as the set

$$C(a) = \{ g \in G : g^{-1}ag = a \}.$$

Furthermore, the center of $G$ is defined as the set

$$Z(G) = \{ g \in G : gx = xg \text{ for all } x \in G \}.$$

(a) Show that $C(a)$ is a subgroup of $G$. [3]

(b) Show that $C(a) = G$ if and only if $a \in Z(G)$. [3]
(Problem 2 - Cont’d)
4. Answer the following questions.

(a) Let $G = \langle g \rangle$ where $|g| = n$. Show that $G = \langle g^k \rangle$ if and only if $gcd(k, n) = 1$. [5]

(b) For the cyclic group $G = \langle g \rangle$ with $|g| = 12$, list all the group generators. [3]
(Problem 4 - Cont’d)
5. Consider the group $G = \{ a, b, c, d, f, g \}$.

(a) Fill its Cayley table below.

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$f$</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$c$</td>
<td>$f$</td>
<td>$c$</td>
<td>$f$</td>
<td>$d$</td>
<td>$g$</td>
</tr>
<tr>
<td>$b$</td>
<td>$f$</td>
<td>$c$</td>
<td>$a$</td>
<td>$f$</td>
<td>$d$</td>
<td>$g$</td>
</tr>
<tr>
<td>$c$</td>
<td>$a$</td>
<td>$f$</td>
<td>$c$</td>
<td>$a$</td>
<td>$f$</td>
<td>$d$</td>
</tr>
<tr>
<td>$d$</td>
<td>$c$</td>
<td>$a$</td>
<td>$f$</td>
<td>$c$</td>
<td>$a$</td>
<td>$f$</td>
</tr>
<tr>
<td>$e$</td>
<td>$d$</td>
<td>$g$</td>
<td>$c$</td>
<td>$d$</td>
<td>$g$</td>
<td>$e$</td>
</tr>
</tbody>
</table>

(b) Which element of $G$ is the unity? 

(c) If $H$ is a subgroup of $G$, what are the possible values of its order $|H|$? Justify your answer.
(Problem 5 - Cont’d)
6. Answer the following questions.

(a) State the definition of a group isomorphism.  

(b) Let \( \alpha : G \to G' \) be a group homomorphism. Define its kernel and show that it is a subgroup of \( G \).
(Problem 6 - Cont’d)
7. Let $\alpha : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$ be the exponential function $\alpha(x) = e^x$ for all $x \in \mathbb{R}$, where $\mathbb{R} \setminus \{0\}$ is the set of non-zero real numbers.

(a) Is $\alpha$ onto? Is it one-to-one? Justify your answers. [2]

(b) Is $\alpha$ a homomorphism? If so, show it and determine its kernel. [3]

(c) Is $\alpha$ an isomorphism? Justify your answer. [1]
8. Consider a \((6,3)\) binary linear code with generator matrix
\[
G = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 \\
\end{bmatrix}.
\]

(a) List all the code’s possible messages with the corresponding codewords. [2]

(b) Evaluate the code’s error detection/correction capabilities. [2]

(c) Assume that the following binary stream is received:

```
011110 011111 011011 110110 111101 000110 000101 100110.
```

Use syndrome decoding to decode the above stream and find the corresponding decoded binary message. [4]

(d) Assuming that the binary message 3-tuples are equivalent to the following letters

```
```

find out the meaning of the message decoded in part (c). [2]
(Problem 8 - Cont’d - i)
(Problem 8 - Cont’d - ii)