Queen’s University
Department of Mathematics and Statistics
MTHE 217 - Algebraic Structures with Applications

Final Exam
December 17, 2016
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Student Number: _______________

Directions

• Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.

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• Total points = 60. Duration = 3 hours.

• Closed Book, Closed Notes, No Calculators Permitted.

• Answers should be put in the space provided.

• Be lucid and neat. Justify all your answers.

• Good Luck!

Marks: Please do not write in the space below.

Problem 1 [8] Problem 5 [8]
Problem 2 [8] Problem 6 [8]
Total: [60]
1. Answer the following questions.

(a) Solve the congruence $4x \equiv 3 \pmod{7}$. [4]

(b) Find the remainder when $7^{311}$ is divided by 5. [4]
2. Let \( a \) and \( b \) be two positive integers, and let \( d = \gcd(a, b) \). Recall that an integer \( l \) is said to be the \textit{least common multiple of} \( a \) and \( b \) – written as \( l = \text{lcm}(a, b) \) – if (i) \( l > 0 \), (ii) \( a \mid l \) and \( b \mid l \), and (iii) for any integer \( n \) such that \( a \mid n \) and \( b \mid n \), we have that \( l \mid n \).

Show that

\[
l = \frac{ab}{d}.
\]
(Problem 2 - Cont’d)
3. Given that $G = \{a, b, c, d, f\}$ is a group, fill its Cayley table given below.

\[
\begin{array}{|c|cccc|}
\hline
& a & b & c & d & f \\
\hline
a & b & & f & & \\
b & c & d & f & & \\
c & d & a & b & & \\
d & & & & & \\
f & & & f & & \\
\hline
\end{array}
\]
(Problem 3 - Cont’d)
4. Answer the following questions.

(a) A group $G$ has subgroups of order 4 and 10 and satisfies $|G| < 50$. What can you conclude about $|G|$? [3]

(b) Assume that $G$ is a group with a subgroup $H$ such that $|G| < 45$, $|H| > 10$ and $|G : H| > 3$, where $|G : H|$ is the index of $H$ in $G$ (i.e., the number of distinct right-cosets of $H$). Find $|G|$, $|H|$ and $|G : H|$. [3]
(Problem 4 - Cont’d)
5. Consider two cyclic groups $G = \langle g \rangle$ and $H = \langle h \rangle$, where $|g| = m$ and $|h| = n$ such that $gcd(m, n) = 1$. Show that the direct product group $G \times H$ is cyclic.
(Problem 5 - Cont’d)
6. Let

\[ \text{SO}(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R}, a^2 + b^2 = 1 \right\} \]

be the special orthogonal set.

(a) Show that \( \text{SO}(2, \mathbb{R}) \) is a subgroup of \( \text{GL}(2, \mathbb{R}) \), the linear group of all \( 2 \times 2 \) real-valued matrices with non-zero determinant. [2]

(b) Let \( \phi: \text{SO}(2, \mathbb{R}) \rightarrow \mathbb{C}^* \) be the function given by

\[ \phi \left( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right) = a + ib, \]

where \( \mathbb{C}^* \) is the group of non-zero complex numbers and \( i \) is the imaginary unit \( (i^2 = -1) \).

Show that \( \phi \) is a 1-1 group homomorphism and find its image. [6]
(Problem 6 - Cont’d)
7. Show that the composition of two group isomorphisms is a group isomorphism. [6]
8. Consider a \((6,3)\) binary linear code \(C\) with parity-check matrix

\[
H = \begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
\]

(a) Find the generator matrix \(G\) of the code \(C\). [1]

(b) List all the code’s possible message words with the corresponding codewords and evaluate the code’s error detection/correction capabilities. [3]

(c) Assume that the following binary stream is received:

\[
011000 \ 111101 \ 010101 \ 110111 \ 101111 \ 001000 \ 000110 \ 000101.
\]

Use syndrome decoding to decode the above stream and find the corresponding decoded binary message. [4]

(d) Assuming that the binary message tuples represent the following letters

\[
000: \text{blank}, \ 001: S, \ 010: 1, \ 011: P, \ 100: 9, \ 101: E, \ 110: L, \ 111: A,
\]

find out the meaning of the message decoded in part (c). [2]
(Problem 8 - Cont’d - i)
(Problem 8 - Cont’d - ii)