16.5

Q19. \text{ Soln:} \\
\text{Eqn of sphere of radius } K \text{ in Cartesian: } x^2 + y^2 + z^2 = K^2 \\
\text{in Cylindrical: } \rho^2 + z^2 = K^2 \\
\text{(Put } x = \rho \cos \theta, \ y = \rho \sin \theta) \\
\text{Using this, volume} \\
\iiint_W \rho \, dV = \iiint_0^2 \rho \, dz \, dr \, d\theta \\
\text{(Note that } \rho \text{ is always } \geq 0) \\
\text{Q20: Example done in class} \\

Q21. \ W \text{ is a solid cone.} \\
\text{In Cartesian co-ord: one familiar cone is given by} \\
z^2 = x^2 + y^2. \\
\text{What does this give in cylindrical co-ord?} \\
\text{We get } z^2 = \rho^2 \text{ i.e. } z = \rho \text{ or } -\rho \text{ depending on whether } z \text{ is } +ve \text{ or } -ve. \\
\text{The equation to expect is some linear dependence between } \rho \text{ and } z. \\
\text{Here, } z = 0 \text{ when } \rho = 0 \text{ (base of the cone)} \\
\text{and } z = 4 \text{ when } \rho = 2 \text{ (top of the cone)} \\
\text{So the equation is } z = 2\rho. \\
\text{As we want } dz \text{ first, let us fix some } \rho, \theta \text{.} \\
\text{Then for this fixed } \rho, \theta, \text{ } z \text{ goes from } 2\rho \text{ to } 4. \\
\text{Done with } z. \\
\rho \text{ goes from } 0 \text{ to } 2 \\
\theta \text{ goes from } 0 \text{ to } 2\pi \text{.}
\[ \int_0^{2\pi} \int_0^a \int_{f(r, \theta, z)}^4 \, r \, dz \, dr \, d\theta \]

Q22

Want \[ \int_0^4 \int_0^\sqrt{4 - \theta^2} \int_0^\pi g(s, \phi, \theta) \, s^2 \sin \phi \, ds \, d\phi \, d\theta \]

Fix \( \phi \) & \( \theta \).

The top of the cone is \( z = 4 \).

In spherical coordinates, this gives \( s \cos \phi = 4 \).

\[ \therefore \text{For fixed } \theta \text{ & } \phi, \]

\( s \) goes from 0 to \( \frac{4}{\cos \phi} \).

To find the range of \( \phi \): consider:

\[ \tan \phi = \frac{1}{2} \text{ for any point } (s, \phi, \theta) \text{ on the surface of the cone} \]

\[ \therefore \phi \text{ goes from } 0 \text{ to } \tan^{-1} \left( \frac{1}{2} \right) \]

\( \theta \) goes from 0 to \( \pi \).

\& as \( W \) has "radial" symmetry around the origin.

\[ \int_0^{2\pi} \int_0^{\tan^{-1}(1/2)} \int_{\cos \phi}^{\pi} g(s, \phi, \theta) \, s^2 \sin \phi \, ds \, d\phi \, d\theta. \]
Q23. As seen in Q21, equation of the cone is
\[ z = 2\sqrt{x^2 + y^2} \]
Fix \( x, y \): then \( z \) goes from \( 2\sqrt{x^2 + y^2} \) to 4.

on \( xy \) plane:
\[ x^2 + y^2 = 4 \]
\[ \int_{-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y, z) \, dz \, dy \, dx. \]

Q24. a)

One eighth of a sphere:
This is a unit sphere
\[ x^2 + y^2 + z^2 = 1 \]
In this \( 1/8 \)th, the \( x, y \) co-ord are +ve while \( z \) is negative
\[ \iiint_{W} f \, dv = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f \, dz \, dy \, dx. \]
Now we are done with $z$. Consider the xy plane. Projection of $W$ into xy plane is a quarter of the circle $x^2 + y^2 = 1$.

Fix $a$: $y$ goes from $0$ to $\sqrt{1-x^2}$

Finally, $x$ goes from $0$ to $1$

\[
\iiint_W dv = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{-\sqrt{1-x^2-y^2}} dz \, dy \, dx
\]

b) Equation of a sphere in cylindrical co-ordinates is $r^2 + z^2 = 1$.

Fix $r, \theta$. Then $z$ goes from $-\sqrt{1-r^2}$ to $0$.

For limits on $r, \theta$, consider the projection in the xy plane:

Fix $\theta$. Then $r$ goes from $0$ to $1$.

Finally, $\theta$ goes from $0$ to $\pi/2$.

\[
\iiint_W dv = \int_0^1 \int_0^{\pi/2} \int_0^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta
\]

c) Equation of the unit sphere in spherical co-ordinates is $\rho = 1$.

Fix an angle $\theta$ with positive $x$ axis & $\phi$ with positive $z$ axis. Then $\rho$ goes from $0$ to $1$.

Fix $\theta$. Then $\phi$ goes from $\pi/2$ to $\pi$.

Finally, $\theta$ goes from $0$ to $\pi/2$. 
\[ \int dv = \int_0^{\pi/2} \int_0^1 \int_0^1 r^2 \sin \phi \, ds \, dp \, d\phi \]

Q26.

We know that the equation of a cone is of the form
\[ z = c \sqrt{x^2 + y^2} \] 
for some \( c \in \mathbb{R} \).

In order to find \( c \) for this cone, let's try to find a point on the surface of this cone.

\[ \tan \frac{\pi}{4} = \frac{AB}{OA} \]
\[ \tan \frac{\pi}{4} = \frac{\sqrt{2}}{OA} \]
\[ \therefore AB = OA \quad (\because \tan \frac{\pi}{4} = 1) \]
\[ \therefore A \Omega B \text{ has length } \sqrt{2} \]
\[ \therefore B = (0, \frac{1}{2}, \frac{1}{2}) \]

Plugging this into \( \text{ii} \), gives
\[ \frac{1}{\sqrt{2}} = c \sqrt{\frac{1}{2}} \Rightarrow c = \pm 1 \]

However, as \( z \) is positive for our cone, we must have \( c = 1 \).

\[ \therefore \text{Eqn is } x^2 + y^2 = z^2 \text{ or } z = \sqrt{x^2 + y^2} \]

a). Fix \( x \) and \( y \).

Then \( z \) goes from \( \sqrt{x^2 + y^2} \) to \( \frac{1}{\sqrt{2}} \).
Project on xy-plane to get $x^2 + y^2 = \frac{1}{2}$.

The plane $z = \frac{1}{\sqrt{2}}$ intersects the cone
$$z = \sqrt{x^2 + y^2}$$
in
$$\sqrt{x^2 + y^2} = \frac{1}{\sqrt{2}}$$
i.e. $x^2 + y^2 = \frac{1}{2}$.

\[\therefore\text{ when we project the cone on the xy plane, we get the circle } x^2 + y^2 = \frac{1}{2} \text{ of radius } \frac{1}{\sqrt{2}}\]

![Figure 1](image)

Fix $\alpha$. Then $y$ goes from $-\sqrt{\frac{1}{2} - x^2}$ to $\sqrt{\frac{1}{2} - x^2}$.

Finally $x$ goes from $-\frac{1}{\sqrt{2}}$ to $\frac{1}{\sqrt{2}}$.

\[\therefore \int_{V} \int_{x^2 + y^2 = \frac{1}{2}} dz \, dy \, dx\]

b) Eqn of the cone in cylindrical co-ord is $z = r$.

Fix $r, \theta$. As similar to a), can show that $z$ goes from $r$ to $\frac{1}{\sqrt{2}}$. For the circle of radius $\frac{1}{\sqrt{2}}$ on the xy plane (Figure 1 above), for fixed $\theta$,

$r$ goes from $0$ to $\frac{1}{\sqrt{2}}$. Finally $\theta$ goes from $0$ to $\pi$.
We get
\[ \int_{\Omega} dv = \int_{0}^{2\pi} \int_{0}^{\frac{1}{\sqrt{2}}} \int_{0}^{\frac{1}{\sqrt{2}}} r \, dr \, dz \, d\theta \]

C) Equation of this cone in spherical co-ord is simply \( \phi = \frac{\pi}{4} \). (Check this.)
Fix some \( \theta \) & \( \phi \).

Eqn of this flat part is \( z = \frac{1}{\sqrt{2}} \)
\[ \implies 8 \cos \phi = \frac{1}{\sqrt{2}} \]
Then \( s \) goes from \( 0 \) to \( \frac{1}{\sqrt{2}} \cos \phi \).

range of \( \phi \): Clearly \( \phi \) goes from \( 0 \) to \( \frac{\pi}{4} \).
range of \( \theta \): \( \theta \) goes from \( 0 \) to \( 2\pi \).

\[ \int_{\Omega} dv = \int_{0}^{2\pi} \int_{0}^{\frac{1}{\sqrt{2}}} \int_{0}^{\frac{1}{\sqrt{2}} \cos \phi} s^2 \sin \phi \, ds \, d\phi \, d\theta \]

\( \phi \leq 0 \) can be done similarly to \( \phi \geq 0 \).

The flat part on top is replaced with part of a sphere.
136 Find: Volume between the cone \( z = \sqrt{x^2+y^2} \) & the plane \( z = 10+x \) above the disk \( x^2+y^2 \leq 1 \).

Sln. we have the following surfaces:

1. \( z = \sqrt{x^2+y^2} \)
2. \( z = 10+x \)

on the disk \( x^2+y^2 \leq 1 \), 1 gives \( z \leq 1 \).

\[
x^2+y^2 = 1 \Rightarrow \frac{x^2}{1} = \frac{y^2}{1} \Rightarrow -1 \leq x \leq 1 & -1 \leq y \leq 1 \\
\text{on the disc}
\]

This can also be seen by etching the disc \( D = x^2+y^2 \leq 1 \)

This means that on the disc \( D \), 2 gives \( z \geq 9 \).

Hence 2 lies above 1 & in the region \( D \).

As 1 is a cone, we try to use cylindrical co-ord.

In cylindrical co-ord,

1. \( z \) is \( z = r \)
2. \( z \) is \( z = 10 + r \cos \theta \)

Fix \( r, \theta \). Then \( z \) goes from \( r \) to \( 10 + r \cos \theta \) (remembering that 2 lies above 1 & in our region \( D \)).

Project for \( r, \theta \) we consider the \( xy \) plane.

so we only have to consider the region \( D \).
For \( D \), clearly \( r \) goes from 0 to 1
\( \theta \) " " " 0 to \( 2\pi \).

\[
\text{Volume} = \int_{0}^{2\pi} \int_{0}^{1} \int_{r}^{\sqrt{10 + r\cos \theta}} r \, dz \, dr \, d\theta
\]
\[
= \int_{0}^{2\pi} \int_{0}^{1} r \left( 10 + r\cos \theta - r \right) \, dr \, d\theta
\]
\[
= \int_{0}^{2\pi} \left( 5r^2 + r^3 \cos \theta - \frac{r^3}{3} \right)_{0}^{1} \, d\theta
\]
\[
= 5(2\pi) + \frac{1}{3} \int_{0}^{2\pi} \cos \theta \, d\theta - \frac{1}{3}(2\pi)
\]
\[
= \sin 2\pi - \sin 0
\]
\[
= 0
\]
\[
= (5 - \frac{1}{3})2\pi = \frac{28\pi}{3}
\]

Note
This could also have been done using Cartesian coord. We would have obtained

\[
\text{Volume} = \int_{0}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{1-x^2}^{10+x} dz \, dy \, dx.
\]

However, the integral is hard to evaluate!
Q37.

The cone \( x^2 = \sqrt{y^2+z^2} \) & the sphere \( x^2+y^2+z^2 = 4 \) intersect in:

\[ y^2+z^2 + y^2+z^2 = 4 \text{ i.e. } y^2+z^2 = 2 \]

This is a circle of radius \( \sqrt{2} \).

We want to find the shaded volume.

If we rotate everything anticlockwise by \( \frac{\pi}{2} \), then this problem is the same as Q37 of 16.4 & the volume needed is the same as the volume of an ice-cream cone bounded by the hemisphere \( z = x^2+y^2+z^2 = 4 \) & an hemisphere \( z = \sqrt{4-x^2-y^2} \) and the cone \( z = \sqrt{x^2+y^2} \).
We can either do this exactly as Q27 of 16.4 or in a simpler way as follows:

Choose cylindrical co-ordinates.

Fixed volume $\int \int \int W \, r \, dz \, dr \, d\theta$

Fixed $r, \theta$: Then $z$ goes from $r$ to $\sqrt{4-r^2}$

For $r, \theta$: We look at the projection on the $xy$ plane.

This gives the circle $x^2 + y^2 = 2$

Fix $\theta$: $r$ goes from 0 to $\sqrt{2}$

$\theta$ goes from 0 to $\pi$.

Thus:

$\text{Volume} = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$

$= \int_0^{2\pi} \int_0^{\sqrt{2}} \left( \sqrt{4-r^2} - r \right) r \, dr \, d\theta$

$= \int_0^{2\pi} \int_0^{\sqrt{2}} \frac{\sqrt{4-r^2} - r^2}{2} \, dr \, d\theta$

$= \int_0^{2\pi} \left[ \frac{1}{2} (4r - r^3) \right]_0^{\sqrt{2}} \, d\theta$

$= \int_0^{2\pi} \left( \frac{2}{2} - \frac{1}{2} \right) \, d\theta$

$= \frac{1}{2} \cdot 2\pi$

$= \pi$
\[ = \int_0^{2\pi} \int_0^{\sqrt{4-r^2}} r \, dr \, d\theta = -2\frac{\sqrt{2}}{3} (4\pi) \]

To evaluate \( I = \int_0^{\sqrt{4-r^2}} r \, dr \), we use substitution.

Let \( t = 4 - r^2 \)

Then \( \frac{dt}{dr} = -2r \Rightarrow -\frac{dt}{2} = r \, dr \)

\[ \Rightarrow \quad t = 4 - 2 = 2 \]

\[ r = \sqrt{2} \Rightarrow t = 4 - 2 = 2 \]

\[ \therefore \quad I = \int_2^4 \frac{dt}{2} = \frac{1}{2} \int_2^4 t \, dt = \frac{1}{2} \left[ \frac{t^{3/2}}{3/2} \right]_2^4 = \frac{1}{3} \left( (2^3 - \frac{2}{3}) \right) \]

\[ \therefore \quad \text{Volume} \]

\[ = \frac{8\pi}{3} \left( 2^3 - \sqrt{2} \right) - \frac{2\pi}{3} \left( 2\sqrt{2} \right) \]

\[ = \frac{8\pi}{3} \left( 2^3 - 4\sqrt{2} \right) = \frac{16\pi}{3} \left( 1 - \frac{\sqrt{2}}{2} \right) = \frac{16\pi}{3} \left( 1 - \frac{1}{\sqrt{2}} \right) \]

\( \text{Q48. Solution} \)

Equation of the sphere: \( z^2 + r^2 = 5^2 \)

Fix \( r, \theta \): Then \( z \) goes from \( -\sqrt{5^2 - r^2} \) to \( \sqrt{5^2 - r^2} \)

Project on \( xy \)-plane: we get \( R \)

\( r \) goes from 1 to 5

(Instead of 0 to 5.)

\( \theta \) goes from 0 to \( 2\pi \).
\[ \text{Volume} = \int_0^{2\pi} \int_0^\pi \int_{\sqrt{5}^2 - r^2}^{\sqrt{5}^2 - r^2} r \, dz \, dr \, d\theta \]

Evaluate! Final answer = \( 64\sqrt{6} \pi \).

Q54.

a) Density \( S \) is a linear function of radius \( R \).

So \( S = mR + c \) for some \( m, c \in \mathbb{R} \).

At \( R = 7 \), we have \( S = 11 \).

At \( R = 6 \), \( S = 9 \).

\[ \begin{align*}
&1. 9 = 11 - 7m + c \quad \text{solve to get} \\
&2. 9 = 6m + c \quad m = 2, \quad c = -3
\end{align*} \]

\[ \therefore S = 2R - 3 \]

b) Mass of the shell \( W = \int W \, dV \) (where Mass = density \times volume).

As \( S \) is in terms of \( R \), it makes sense to use spherical co-ordinates.

The shell still has spherical symmetry, so we expect \( \phi, \theta \) to go over their full range.

\( R \) should go only from 6 to 7. (why?)

\[ \text{Volume} = \int_0^{2\pi} \int_0^\pi \int_0^7 S(R) \, R^2 \sin \phi \, d\phi \, d\theta \]

\[ = \int_0^{2\pi} \int_0^\pi \int_0^7 (2R - 3) \, R^2 \sin \phi \, d\phi \, d\theta \]

Evaluate! Final answer = \( 1702\pi \).