Problem 19: Find $\int_C (\sin(x^2) + \cos(y)) \hat{i} + (\sin(y^2) + e^x) \hat{j} \cdot d\vec{r}$, where $C$ is the square of side 1 in the first quadrant of the $xy$-plane, with one vertex at the origin and sides along the axes, and oriented counterclockwise when viewed from above.

**Note:** As stated in the announcements on the webpage, there is a typo in the textbook and the plus sign in the first term is missed in the textbook.

**Solution.** This integral is of the form $\int_C \vec{F} \cdot d\vec{r}$, with $\vec{F} = (\sin(x^2) + \cos(y)) \hat{i} + (\sin(y^2) + e^x) \hat{j}$. Since $\vec{F}$ is continuous with continuous partial derivatives and is defined everywhere in the interior of $C$ (no “holes” in the domain), we can apply Green’s theorem. This gives us that the required integral is

$$\int_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA,$$

where $R$ is the region enclosed by $C$. As $\partial F_2 / \partial x = e^x$ and $\partial F_1 / \partial y = -\sin y$, we obtain that the required integral is

$$\int_0^1 \int_0^1 e^x + \sin y dy dx = \int_0^1 (e^1 - e^0) + \sin y dy = e - 1 + (\cos y)|_0^1.$$

This gives $e - \cos 1$. □

Problem 27. Calculate the area of the region within the ellipse $x^2/a^2 + y^2/b^2 = 1$, parameterized by $x = a \cos t$, $y = b \sin t$, for $0 \leq t \leq 2\pi$.

**Solution.** We would like to use the identity

(1) \[ \int_C x \hat{j} \cdot d\vec{r} = \int_R dA, \]

where, $C$ is a closed curve and $R$ is the region enclosed by it. This is stated in Problem 26 of the same section and can be proved by an application of Green’s theorem. In fact one can prove more generally that:

$$\int_C x \hat{j} \cdot d\vec{r} = \int_C -y \hat{i} \cdot d\vec{r} = \frac{1}{2} \int_C x \hat{j} - y \hat{i} \cdot d\vec{r} = \int_R dA,$$

using Green’s theorem.
Using (1), we have that the area \( A \) of the ellipse is given by

\[
A = \int_C \mathbf{x}^\mathbf{\hat{j}} \cdot d\mathbf{r},
\]

where \( \mathbf{r}(t) = a \cos t \mathbf{\hat{i}} + b \sin t \mathbf{\hat{j}} \), with \( 0 \leq t \leq 2\pi \). As \( \mathbf{r}'(t) = -a \sin t \mathbf{\hat{i}} + b \cos t \mathbf{\hat{j}} \), we have

\[
A = \int_0^{2\pi} a \cos t \mathbf{\hat{j}} \cdot (-a \sin t \mathbf{\hat{i}} + b \cos t \mathbf{\hat{j}}) \, dt
\]

\[
= \int_0^{2\pi} ab \cos^2 t = \frac{ab}{2} \int_0^{2\pi} (1 + \cos 2t)
\]

\[
= \pi ab + \frac{ab}{2} \int_0^{2\pi} \cos 2t
\]

Note that here we have used the trigonometric identity

\[
\cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1.
\]

In the final expression above, the integral \( \int_0^{2\pi} \cos 2t \) is zero because of the periodicity of cosine. This gives that the area of the ellipse is \( A = ab\pi \). \( \square \)

**Problem 33.**

(a) By finding potential functions, show that each of the vector fields \( \mathbf{F}, \mathbf{G}, \mathbf{H} \) is a gradient field on some domain (not necessarily the whole plane).

(b) Find the line integrals of \( \mathbf{F}, \mathbf{G}, \mathbf{H} \) around the unit circle in the \( xy \)-plane, centered at the origin, and traversed counterclockwise.

(c) For which of the three vector fields can Green’s Theorem be used to calculate the line integral in part (b)? Why or why not?

\[
\mathbf{F} = y \mathbf{\hat{i}} + x \mathbf{\hat{j}}, \quad \mathbf{G} = \frac{y \mathbf{\hat{i}} - x \mathbf{\hat{j}}}{x^2 + y^2}, \quad \mathbf{H} = \frac{x \mathbf{\hat{i}} + y \mathbf{\hat{j}}}{(x^2 + y^2)^{1/2}}.
\]

**Solution.** (a) To find a potential function for \( F \), we want to find a function \( f(x,y) \) such that

\[
\partial f/\partial x = F_1 = y, \quad \partial f/\partial y = F_2 = x.
\]

Solving the first equation gives \( f = xy + C(y) \), where \( C \) is some function of \( y \). From this, we obtain \( \partial f/\partial y = x + C'(y) \). Comparing this with the second equation in (2), we see that \( C'(y) = 0 \), so that \( C \) is a constant independent of \( y \). Thus, we have obtained : \( f(x,y) = xy + c \) is a potential function for \( \mathbf{F} \).

To find a potential function for \( G \), we want to find a function \( g(x,y) \) such that

\[
\partial g/\partial x = y/(x^2 + y^2), \quad \partial g/\partial y = -x/(x^2 + y^2).
\]
Solving the first equation gives
\[ g(x, y) = y \int \frac{1}{x^2 + y^2} dx = \frac{y}{y^2} \int \frac{1}{(x/y)^2 + 1} dx. \]

We make the change of variable \( u = x/y \) so that \( ydu = dx \). Then, \( g(x, y) = \)
\[ \frac{y}{y^2} \int \frac{1}{1 + u^2} ydu = \int \frac{1}{1 + u^2} du = \arctan(u) + C(y). \]

Thus, \( g(x, y) = \arctan(x/y) + C(y) \), where \( C \) is some function of \( y \). From this, we obtain \( \partial g/\partial y = C'(y) + \partial(\arctan(x/y))/\partial y \). To simplify the second expression, we must use the chain rule.
\[ \frac{\partial(\arctan(x/y))}{\partial y} = \frac{\partial(\arctan(x/y))}{\partial (x/y)} \frac{\partial (x/y)}{\partial y} = \frac{1}{1 + (x/y)^2} \frac{-x}{y^2}. \]

Note that we have used \( d(\arctan t)/dt = 1/(1+t^2) \). Thus,
\[ \partial g/\partial y = C'(y) + \frac{1}{1 + (x/y)^2} \frac{-x}{y^2} = C'(y) + \frac{-x}{x^2 + y^2}. \]

Comparing this with the second equation in (3), we see that \( C'(y) = 0 \), so that \( C \) is a constant independent of \( y \). Thus, we have obtained: \( g(x, y) = \arctan(x/y) + c \) is a potential function for \( \vec{G} \).

To find a potential function for \( \vec{H} \), we want to find a function \( h(x, y) \) such that
\[ \partial h/\partial x = x/(x^2 + y^2)^{1/2}, \quad \partial h/\partial y = y/(x^2 + y^2)^{1/2}. \]

Solving the first equation gives
\[ h(x, y) = \int \frac{x}{(x^2 + y^2)^{1/2}} dx. \]

We make the change of variable \( u = x^2 + y^2 \) so that \( du = 2xdx \). Then, \( h(x, y) \) equals
\[ \frac{1}{2} \int u^{-1/2} du = u^{1/2} = (x^2 + y^2)^{1/2} + C(y), \]

where \( C \) is some function of \( y \). From this, after some work, we obtain \( \partial h/\partial y = C'(y) + y/(x^2 + y^2)^{1/2} \). Comparing this with the second equation in (4), we see that \( C'(y) = 0 \), so that a potential function for \( \vec{H} \) is given by \( h(x, y) = (x^2 + y^2)^{1/2} + c \).

(b) We will find the required line integrals in the straightforward manner by parametrizing the unit circle \( C \) traversed counterclockwise as \( \vec{r}(t) = \cos t \hat{i} + \sin t \hat{j}, 0 \leq t \leq 2\pi \). Then \( \vec{r}'(t) = -\sin t \hat{i} + \cos t \hat{j} \).
The line integral of $\vec{F}$ along $C$ is
\[\int_0^{2\pi} \vec{F}(\cos t, \sin t) \cdot \vec{r}'(t) \, dt = \int_0^{2\pi} \cos^2 t - \sin^2 t \, dt = \int_0^{2\pi} \cos(2t) \, dt,\]
using the double angle formula for cosine: $\cos 2t = \cos^2 t - \sin^2 t$. Due to the periodicity of cosine, this integral is 0.

The line integral of $\vec{H}$ along $C$ is easily seen to be zero because $\vec{H}$ is nothing but $\vec{r}/||\vec{r}||$ - a vector normal to the circle (perpendicular to the tangent to the circle) at any point.

We now consider the vector field $\vec{G}$. One can check that the line integral of $\vec{G}$ along $C$ is
\[\int_0^{2\pi} \vec{G}(\cos t, \sin t) \cdot \vec{r}'(t) \, dt = -\int_0^{2\pi} \cos^2 t + \sin^2 t \, dt = -\int_0^{2\pi} dt = -2\pi.\]
(c) Green’s theorem if applicable, should give 0 for all the line integrals of the previous part. However, this theorem cannot be used to calculate the line integral along $C$ for $\vec{G}$ and $\vec{H}$ as these vector fields are not defined at the origin $(0,0)$ which lies inside $C$. 

□