Problem 35: Find the flux of \( \mathbf{F} = 5\mathbf{i} + 7\mathbf{j} + z\mathbf{k} \) through a closed cylinder \( S \) of radius 3 centered on the z-axis, with \(-2 \leq z \leq 2\), and oriented outward.

Solution. The surface \( S \) is made up of three surfaces - the top, bottom and curved part of the cylinder. The formula for flux through a cylindrical surface derived in class (stated in section 19.2) only gives the flux through the curved part of the cylinder (why?). We first evaluate this and then consider the top and bottom of the cylinder.

Flux through the curved surface of the cylinder. We call this surface \( S_1 \). By the formula derived in class,

\[
\text{Flux}_{S_1} = \int_{0}^{2\pi} \int_{-2}^{2} (5\mathbf{i} + 7\mathbf{j} + z\mathbf{k}) \cdot (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) \, 3 \, dz \, d\theta
\]

\[
= 3 \int_{0}^{2\pi} \int_{-2}^{2} 5 \cos \theta + 7 \sin \theta \, dz \, d\theta
\]

\[
= 12(5 \sin \theta - 7 \cos \theta)|_{0}^{2\pi} = 0,
\]
due to the periodicity of sine and cosine.

Another way to see this is by observing that the curved surface of the cylinder has area vector perpendicular to \( \mathbf{k} \) at every point. This means that only \( \mathbf{i} \) and \( \mathbf{j} \) components of \( \mathbf{F} \) matter for the flux. However, these are constant and the flux of a constant vector field through a closed surface is zero.

Flux through the top of the cylinder. Notice that any point on the top of the cylinder has fixed \( z \) component \( z = 2 \). This means that \( \mathbf{F} \) on the top of the cylinder is given by \( 5\mathbf{i} + 7\mathbf{j} + 2\mathbf{k} \).

Let us call the top of the cylinder as surface \( S_2 \). The area vector of \( S_2 \) is parallel to \( \mathbf{k} \), which means that we can write \( d\mathbf{A} = (dA)\mathbf{k} \). With all this information, the flux through the top is given by

\[
\text{Flux}_{S_2} = \int_{S_1} (5\mathbf{i} + 7\mathbf{j} + 2\mathbf{k})(\mathbf{k})dA = 2 \int_{S_2} dA
\]

The integral above is simply the area of the top of the cylinder of radius 3, that is \( \pi 3^2 \). Thus, we get \( \text{Flux}_{S_2} = 18\pi \).
Flux through the bottom of the cylinder. Any point on the bottom of the cylinder has fixed $z$ component $z = -2$. This means that $\vec{F}$ on the bottom of the cylinder is given by $5\hat{i} + 7\hat{j} - 2\hat{k}$.

Let us call the bottom of the cylinder as surface $S_3$. The area vector of $S_3$ is parallel to $-\hat{k}$, which means that we can write $d\vec{A} = -(dA)\hat{k}$. With all this information, the flux through the top is given by

$$\text{Flux}_{S_3} = \int_{S_3} (5\hat{i} + 7\hat{j} - 2\hat{k})(-\hat{k})dA = 2 \int_{S_3} dA$$

This gives $\text{Flux}_{S_3} = 18\pi$, as before.

Net flux. The flux of $\vec{F}$ through the closed cylinder given in the problem is now given by

$$\text{Flux}_{S_1} + \text{Flux}_{S_2} + \text{Flux}_{S_3} = 36\pi.$$
This gives $4 \times 6 \times 2\pi$ using the identity $\cos^2 \theta + \sin^2 \theta = 1$. Hence $\text{Flux}_{S_1} = 48\pi$.

**Flux through the top of the cylinder.** We now stop assuming that the cylinder is around the $z$-axis and work with the cylinder as given. Any point on the top of the cylinder has fixed $y$ component $y = 3$. This means that $\vec{F}$ on the top of the cylinder is given by $x\hat{i} + 3\hat{j} + z\hat{k}$.

Let us call the top of the cylinder as surface $S_2$. The area vector of $S_2$ is parallel to $\hat{j}$, which means that we can write $d\vec{A} = (dA)\hat{j}$. With all this information, the flux through the top is given by

$$\text{Flux}_{S_2} = \int_{S_1} ((x\hat{i} + 3\hat{j} + z\hat{k})(\hat{j})dA = 3 \int_{S_2} dA$$

The integral above is simply the area of the top of the cylinder of radius 2, that is $\pi 2^2$. Thus, we get $\text{Flux}_{S_2} = 12\pi$.

**Flux through the bottom of the cylinder.** Any point on the bottom of the cylinder has fixed $y$ component $y = -3$. This means that $\vec{F}$ on the bottom of the cylinder is given by $x\hat{i} - 3\hat{j} + z\hat{k}$.

Let us call the bottom of the cylinder as surface $S_3$. The area vector of $S_3$ is parallel to $-\hat{j}$, which means that we can write $d\vec{A} = -(dA)\hat{j}$. With all this information, the flux through the top is given by

$$\text{Flux}_{S_3} = \int_{S_2} ((x\hat{i} - 3\hat{j} + z\hat{k})(-\hat{j})dA = 3 \int_{S_3} dA$$

This gives $\text{Flux}_{S_3} = 12\pi$, as before.

**Net flux.** Adding all three answers, we get that the flux of $\vec{F}$ through the closed cylinder given is $72\pi$. 

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