Problem 1: Evaluate the line integral \( \int_C y\,dx + x\,dy \), where \( C \) is the parameterized path \( \vec{r}(t) = (t^2, t^3) \), \( 1 \leq t \leq 5 \).

Solution. On the path \( \vec{r}(t) \), we have \( x(t) = t^2 \), \( y(t) = t^3 \), \( dx = 2tdt \), \( dy = 3t^2dt \). Hence the line integral is
\[
\int_1^5 t^3(2tdt) + t^2(3t^2dt) = \int_1^5 5t^4dt = (5^5 - 1).
\]

Problem 2. Consider the parabola \( C \) given by \( x = y^2 + 2 \).

(a) Write an integral for the arc length of the parabola between the points \((3, 1)\) and \((6, -2)\).

(b) Give the parametric equation of the tangent line to the parabola at the point \((2, 0)\).

Solution. We parametrize \( C \) by putting \( y(t) = t \) and \( x(t) = t^2 + 2 \) so that the equation of the parabola holds for all values of \( t \). This gives \( dx/dt = 2t \), \( dy/dt = 1 \).

(a) The points \((3, 1)\), \((6, -2)\) are reached at \( t = 1, -2 \) respectively. Arc length is given by
\[
\int_{-2}^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_{-2}^1 \sqrt{4t^2 + 1} \, dt
\]
Note that here we took the lower limit to be \( t = -2 \) and upper limit to be \( t = 1 \), as arc length must be positive.

(b) The point \((2, 0)\) is reached at \( t = 0 \) by the path \( \vec{r}(t) = (t^2 + 2)i + t\vec{j} \). We first find the velocity vector at this point: \( \vec{v}(t) = 2t\vec{i} + \vec{j} \). Hence the required tangent line is parallel to \( \vec{v}(0) \) and passes through the point \((2, 0)\). This gives the parametric equation
\[
\vec{r}(t) = (2, 0) + t\vec{v}(0) = 2\vec{i} + t\vec{j}
\]
for the tangent line.
Problem 3. Evaluate the integral
\[\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{1+x^2}} e^{-(x^2+y^2)} \, dy \, dx.\]

Solution. The symmetry of the integrand and limits suggests that we should use polar co-ordinates. If we sketch the region of integration, we find that it is the right half of the interior of the unit circle \(x^2+y^2 = 1\). This region can be described in polar co-ordinates by \(0 \leq r \leq 1\) and \(-\pi/2 \leq \theta \leq \pi/2\). However, as range of \(\theta\) is \([0, 2\pi]\), we should divide this region into two parts - the top half with \(0 \leq \theta \leq \pi/2\) and the bottom half, with \(\pi \leq \theta \leq \pi/2\). Hence the integral is
\[\int_{\pi/2}^{\pi/2} \int_0^1 e^{-r^2} r \, dr \, d\theta + \int_{3\pi/2}^{\pi/2} \int_0^1 e^{-r^2} r \, dr \, d\theta.\]

For the inner integral, we use the substitution \(u = r^2\) so that \(du = 2r \, dr\), and \(u\) goes from \(0^2\) to \(1^2\), giving
\[\frac{1}{2} \int_0^1 e^{-u} du = \frac{1}{2}(1 - e^{-1}).\]

Hence,
\[\int_0^{\pi/2} \int_0^1 e^{-r^2} r \, dr \, d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left[ \frac{1}{2} \right] \, d\theta = \frac{\pi}{4}(1 - e^{-1}),\]
and similarly,
\[\int_{3\pi/2}^{\pi/2} \int_0^1 e^{-r^2} r \, dr \, d\theta = \frac{\pi}{4}(1 - e^{-1}).\]

This gives \(\pi(1 - e^{-1})/2\) as the value of the required integral. \(\square\)

Problem 4. Evaluate
\[\int_0^\pi \int_y^\pi \frac{\sin x}{x} \, dx \, dy.\]

Solution. Sketching the region of integration shows that it is the interior of the triangle made by the points \((0, 0), (\pi, 0)\) and \((\pi, \pi)\). As this integral does not seem easy to do as given, we reverse the order of integration. For a fixed \(x\), we see that \(y\) goes from 0 to \(x\). Finally, \(x\) goes from 0 to \(\pi\). Hence the integral is
\[\int_0^\pi \int_0^x \frac{\sin x}{x} \, dy \, dx = \int_0^\pi \frac{\sin x}{x} \, x \, dx = (-\cos x)|_0^\pi = 2.\] \(\square\)