Q 1. The surface of Lake Ontario can be represented by a region in the $xy$-plane such that the lake’s depth (in meters) at the point $(x, y)$ is given by $f(x, y) = 600 - 5x^2y^2$.

(a) Find the gradient vector for $f$. (5 marks)

(b) If you are at the point $(5, 1)$ and started swimming towards $(7, 4)$, find the rate of change of depth. Is the depth increasing or decreasing? (4+1 marks)

Solution. (a) As $\frac{\partial f}{\partial x} = -10xy^2$ and $\frac{\partial f}{\partial y} = -10x^2y$, the gradient vector $\nabla f(x, y)$ is given by

$$\nabla f(x, y) = -10xy^2 \hat{i} + -10x^2y \hat{j} = -10xy(y \hat{i} + x \hat{j}).$$

(b) We need to find the directional derivative of the depth function $f$ at the point $(5, 1)$ in the direction of the vector $\vec{u}$, where

$$\vec{u} = (7\hat{i} + 4\hat{j}) - (5\hat{i} + \hat{j}) = 2\hat{i} + 3\hat{j}.$$  

As $f$ is differentiable, we can use $f_{\vec{u}}(5, 1) = \nabla f(5, 1) \cdot \vec{u}$. As $||\vec{u}|| = \sqrt{13}$, we get

$$f_{\vec{u}}(5, 1) = \frac{1}{\sqrt{13}} (-50\hat{i} + -250\hat{j}) \cdot (2\hat{i} + 3\hat{j}) = -\frac{850}{\sqrt{13}}.$$  

This is the rate of change of depth. As the rate of change is negative, the depth is decreasing.  

Q 2. Suppose that

$$z = f \left( \frac{xy}{x^2 + y^2} \right)$$

is a differentiable function. Let $u = \frac{xy}{x^2 + y^2}$.

(a) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. (4 marks)

(b) What is $\frac{\partial z}{\partial x}$ in terms of $\frac{\partial z}{\partial u}$? What is $\frac{\partial z}{\partial y}$ in terms of $\frac{\partial z}{\partial u}$? Simplify your answers as much as possible. (4 marks)

(c) Show that

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0.$$  

(2 marks)
Solution. (a) Using the quotient rule,
\[
\frac{\partial u}{\partial x} = \frac{y(x^2 + y^2) - xy(2x)}{(x^2 + y^2)^2} = \frac{y^3 - x^2y}{(x^2 + y^2)^2},
\]
\[
\frac{\partial u}{\partial y} = \frac{x(x^2 + y^2) - xy(2y)}{(x^2 + y^2)^2} = \frac{x^3 - y^2x}{(x^2 + y^2)^2}.
\]

(b) As \( z \) is a function of \( u \) and \( u \) is a function of \( x \) and \( y \), using the chain rule and then (a), we get
\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} = \frac{y^3 - x^2y}{(x^2 + y^2)^2} \frac{\partial z}{\partial u},
\]
\[
\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} = \frac{x^3 - y^2x}{(x^2 + y^2)^2} \frac{\partial z}{\partial u}.
\]

(c) From (b),
\[
x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \left( \frac{xy^3 - x^3y + yx^3 - y^3x}{(x^2 + y^2)^2} \right) = 0,
\]
as required.

Q 3. Consider the function
\[
f(x, y) = \begin{cases} \frac{xy^3}{x^4 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}
\]

(a) Find \( f_x(0, 0) \) and \( f_y(0, 0) \).

(b) Show that \( f \) is not continuous at (0, 0).

(c) Can \( f \) be differentiable at (0, 0)? Explain your answer briefly.

Solution. (a) We must use the limit definition,
\[
f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h}.
\]

As \( f(0, 0) = 0 \) by definition and plugging \( x = h, y = 0 \) in the expression for \( f(x, y) \) gives \( f(h, 0) = 0 \) we get
\[
f_x(0, 0) = 0.
\]

Similarly,
\[
f_y(0, 0) = \lim_{h \to 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = 0.
\]
(b) To check continuity at \((0, 0)\), we need to check if
\[
\lim_{h,k \to 0} f(h, k) = f(0, 0),
\]
that is, whether
\[
\lim_{h,k \to 0} \frac{hk^3}{h^4 + k^4} = 0,
\]
as \(f(0, 0) = 0\) by definition. Let’s check the special case \(h = k\). The LHS is then
\[
\lim_{h \to 0} \frac{h^4}{2h^4} = 1/2,
\]
which does not equal 0. As a special case is not satisfied, the general case cannot hold and the function is not continuous at \((0, 0)\).
(c) The function \(f\) cannot be differentiable at \((0, 0)\) because it is not continuous at that point.

\[\square\]

Q 4. Consider the function \(f(x, y) = e^{xy}\) on the region \(R\) given by \(1 \leq x \leq 5, 2 \leq y \leq 3\).

(a) Write down the volume under the graph of \(f\) above the region \(R\) as an integral in terms of \(x\) and \(y\). (Do not evaluate the integral). (5 marks)
(b) Write down the area of the region \(R\) as an integral. Conclude what the value of this integral should be. (3+2 marks)

Solution. (a) The volume is
\[
\int_R f dA = \int_2^3 \int_1^5 e^{xy} dxdy = \int_1^5 \int_2^3 e^{xy} dydx.
\]
(b) The area of \(R\) is given by the integral
\[
\int_R dA = \int_2^3 \int_1^5 dxdy = \int_1^5 \int_2^3 dydx.
\]
The value of this integral should be 4 as the area of the rectangle \(R\) is 4.

\[\square\]