Q 1. Suppose that
\[ z = f \left( \frac{x^2 - y^2}{x^2 + y^2} \right) \]
is a differentiable function. Let \( u = \frac{x^2 - y^2}{x^2 + y^2} \).

(a) Find \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \). (4 marks)

(b) What is \( \frac{\partial z}{\partial x} \) in terms of \( \frac{\partial z}{\partial u} \)? What is \( \frac{\partial z}{\partial y} \) in terms of \( \frac{\partial z}{\partial u} \)? Simplify your answers as much as possible. (4 marks)

(c) Show that
\[ x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0. \]

(2 marks)

**Solution.**

(a) Using the quotient rule,
\[
\frac{\partial u}{\partial x} = \frac{2x(x^2 + y^2) - (x^2 - y^2)(2x)}{(x^2 + y^2)^2} = \frac{4xy^2}{(x^2 + y^2)^2},
\]
\[
\frac{\partial u}{\partial y} = \frac{-2y(x^2 + y^2) - (x^2 - y^2)(2y)}{(x^2 + y^2)^2} = \frac{-4yx^2}{(x^2 + y^2)^2}.
\]

(b) As \( z \) is a function of \( u \) and \( u \) is a function of \( x \) and \( y \), using the chain rule and then (a), we get
\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} = \frac{4xy^2}{(x^2 + y^2)^2} \frac{\partial z}{\partial u},
\]
\[
\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} = \frac{-4yx^2}{(x^2 + y^2)^2} \frac{\partial z}{\partial u}.
\]

(c) From (b),
\[
x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \left( \frac{4x^2y^2 - 4y^2x^2}{(x^2 + y^2)^2} \right) = 0,
\]
as required.

\[ \square \]
Q 2. The surface of Lake Ontario can be represented by a region in the $xy$-plane such that the lake’s depth (in meters) at the point $(x, y)$ is given by $f(x, y) = 300 - x^2y^2$.

(a) Find the gradient vector for $f$. (5 marks)

(b) If you are at the point $(2, 1)$ and started swimming towards $(3, 4)$, find the rate of change of depth. Is the depth increasing or decreasing? (4+1 marks)

Solution. (a) As $\frac{\partial f}{\partial x} = -2xy^2$ and $\frac{\partial f}{\partial y} = -2x^2y$, the gradient vector $\nabla f(x, y)$ is given by

$$\nabla f(x, y) = -2xy^2\hat{i} + -2x^2y\hat{j} = -2xy(y\hat{i} + x\hat{j}).$$

(b) We need to find the directional derivative of the depth function $f$ at the point $(2, 1)$ in the direction of the vector $\vec{u}$, where

$\vec{u} = (3\hat{i} + 4\hat{j}) - (2\hat{i} + \hat{j}) = \hat{i} + 3\hat{j}.$

As $f$ is differentiable, we can use $f_{\vec{u}}(2, 1) = \nabla f(2, 1) \cdot \vec{u}$. As $||\vec{u}|| = \sqrt{10}$, we get

$$f_{\vec{u}}(2, 1) = \frac{1}{\sqrt{10}}(-4\hat{i} + -8\hat{j}) \cdot (\hat{i} + 3\hat{j}) = -\frac{28}{\sqrt{10}}.$$

This is the rate of change of depth. As the rate of change is negative, the depth is decreasing. □

Q 3. Consider the function $f(x, y) = x^2y$ on the region $R$ given by $2 \leq x \leq 4, 5 \leq y \leq 9$.

(a) Write down the volume under the graph of $f$ above the region $R$ as an integral in terms of $x$ and $y$. (Do not evaluate the integral). (4 marks)

(b) Write down the area of the region $R$ as an integral. Conclude what the value of this integral should be. (3+2 marks)

Solution. (a) The volume is

$$\int_R f\,dA = \int_5^9 \int_2^4 x^2y \,dxdy = \int_2^4 \int_5^9 x^2y \,dydx.$$

(b) The area of $R$ is given by the integral

$$\int_R \,dA = \int_5^9 \int_2^4 \,dxdy = \int_2^4 \int_5^9 \,dydx.$$

The value of this integral should be 8 as the area of the rectangle $R$ is 8. □
Q 4. Consider the function

\[ f(x, y) = \begin{cases} 
\frac{x^3 y}{x^4 + y^4}, & (x, y) \neq (0, 0) \\
0, & (x, y) = (0, 0)
\end{cases} \]

(a) Find \( f_x(0, 0) \) and \( f_y(0, 0) \). (4 marks)
(b) Show that \( f \) is not continuous at \( (0, 0) \). (4 marks)
(c) Can \( f \) be differentiable at \( (0, 0) \)? Explain your answer briefly. (2 marks)

**Solution.**

(a) We must use the limit definition,

\[ f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h}. \]

As \( f(0, 0) = 0 \) by definition and plugging \( x = h, y = 0 \) in the expression for \( f(x, y) \) gives \( f(h, 0) = 0 \) we get

\[ f_x(0, 0) = 0. \]

Similarly,

\[ f_y(0, 0) = \lim_{h \to 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = 0. \]

(b) To check continuity at \( (0, 0) \), we need to check if

\[ \lim_{h,k \to 0} f(h, k) = f(0, 0), \]

that is, whether

\[ \lim_{h,k \to 0} \frac{h^3 k}{h^4 + k^4} = 0, \]

as \( f(0, 0) = 0 \) by definition. Let’s check the special case \( h = k \). The LHS is then

\[ \lim_{h \to 0} \frac{h^4}{2h^4} = 1/2, \]

which does not equal 0. As a special case is not satisfied, the general case cannot hold and the function is not continuous at \( (0, 0) \).

(c) The function \( f \) cannot be differentiable at \( (0, 0) \) because it is not continuous at that point.

\[ \square \]