APSC 174 • 2007

Aide-memoire

Elementary Row Operations

- switch two rows
- add a multiple of one row to another
- multiply a row by a non-zero scalar

Row Echelon Form

- a matrix is in *row echelon form* if
  - all zero rows at bottom
  - each leading entry to the right of those above
  - only zeros below leading entry
  - $\begin{bmatrix} 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in row echelon form
- a matrix is in *reduced row echelon form* if it is in row echelon form and
  - each leading entry is a 1
  - each leading 1 is the only non-zero entry in its column
  - $\begin{bmatrix} 0 & 1 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in reduced row echelon form
- *pivot* columns are those columns containing leading entries

Linear Relations on Columns

- $A = [\vec{a}_1 \mid \vec{a}_2 \mid \cdots \mid \vec{a}_n]$
- any linear relation between $\{\vec{a}_1, \ldots, \vec{a}_n\}$ can be expressed as a solution of a system $A\vec{x} = \vec{b}$; $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$ if and only if $A\vec{x} = \vec{b}$
- solutions to the system $A\vec{x} = \vec{b}$ are unchanged by elementary row operations
- $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

the augmented matrix of the system is

$$[A \mid \vec{b}] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_n \end{array}\right]$$

Linear Independence

- $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ are *dependent* if one of them can be written as a linear combination of others
- *independent* = not dependent
- $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ are independent if and only if $[\vec{v}_1 \mid \vec{v}_2 \mid \cdots \mid \vec{v}_k]$ has k pivot columns when put in row echelon form; i.e. every column is a pivot column

Linear Transformations

- $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$
- $T(c\,\vec{v}) = c\,T(\vec{v})$
- $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ is the *standard* basis of $\mathbb{R}^n$
- $A_T = [T(\vec{e}_1) \mid T(\vec{e}_2) \mid \cdots \mid T(\vec{e}_n)]$ is the *standard* matrix of T: $T(\vec{x}) = A_T\vec{x}$
- T = rotation by angle θ in the counter-clockwise direction, $A_T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
- $T = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ is a shear matrix

Transposes

- if A is a $m \times n$ matrix, A^T is the $n \times m$ matrix obtained by making the columns of A the rows of A^T
- $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$

Inverses

- $A^{-1}A = AA^{-1} = I_n$
- A^{-1} is the inverse of A
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$

Calculating Inverses

- if $\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right] \sim \cdots$ $\sim \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & 1 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{array}\right]$ then $B = A^{-1}$

Determinants

- $\det(A) = |A|$ = determinant of the square matrix A
- $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

• cofactor expansion: $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

• A is invertible $\Leftrightarrow \det(A) \neq 0$

• $\det(AB) = \det(A)\det(B)$

• det(triangular matrix) = product of diagonal entries

• $\det(A^T) = \det(A)$

• $\det[\vec{v}_1 \,|\, \vec{v}_2]$ = area of parallelogram with sides $\vec{v}_1$ and $\vec{v}_2$

• $\det[\vec{v}_1 \,|\, \vec{v}_2 \,|\, \vec{v}_3]$ = volume of parallelepiped with sides $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$

Subspaces

• $W \subset \mathbb{R}^n$ is a subspace if

- W is closed under addition
- W is closed under scalar multiplication
- W contains $\vec{0}$

• $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is the set of all linear combinations $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$. $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is the subspace *spanned* by $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$

• $\text{Col}(A)$ is the subspace spanned by the column vectors of A

• $\text{Nul}(A)$ is the subspace of all solutions of the homogeneous system $A\vec{x} = \vec{0}$

• $\text{Row}(A)$ is the subspace spanned by the rows of A

• $\text{Row}(A) = \text{Col}(A^T)$

Bases and Dimension

• $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a basis for a subspace W if $W = \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ and $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is independent; i.e. $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a minimal spanning set of vectors in W

• Basis of $\text{Col}(A)$ is $\{\vec{a}_{i_1}, \vec{a}_{i_2}, \dots, \vec{a}_{i_r}\}$ where $\{i_1, i_2, \dots, i_r\}$ are the pivot columns of A

• Basis of $\text{Row}(A)$ use $\text{Row}(A) = \text{Col}(A^T)$ and then previous construction, or the non-zero rows of row echelon form of A

• Basis of $\text{Nul}(A)$ — find parametric vector form of solutions to $A\vec{x} = \vec{0}$: if $x_{i_1}, x_{i_2}, \dots, x_{i_s}$ are the free variables and the general solution is $x_{i_1}\vec{v}_1 + x_{i_2}\vec{v}_2 + \cdots + x_{i_s}\vec{v}_s$, then $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_s\}$ is a basis for $\text{Nul}(A)$

• any two bases of W have the same number of elements; this number is the *dimension* of W

Rank and Nullity

• $\text{rank}(A) = \dim(\text{Col}(A))$ = n° pivot columns of A

• $\text{nul}(A) = \dim(\text{Nul}(A))$ = n° free variables in system $A\vec{x} = \vec{0}$

• for an $m \times n$ matrix A, $\text{rank}(A) + \text{nul}(A) = n$

• A an $n \times n$ matrix:

$\Leftrightarrow$ A is invertible

$\Leftrightarrow$ $\text{nul}(A) = 0$

$\Leftrightarrow$ $\text{rank}(A) = n$

$\Leftrightarrow$ $\text{Col}(A) = \mathbb{R}^n$

$\Leftrightarrow$ $A\vec{x} = \vec{0}$ has only the trivial solution

$\Leftrightarrow$ $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$

Coordinates

• $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots \vec{b}_k\}$ is a basis for a subspace W

• $\vec{x}$ is a vector in W, $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_k\vec{b}_k$; $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$ are the coordinates of $\vec{x}$ with respect to the basis $\mathcal{B}$

Eigenvalues and Eigenvectors

• $A\vec{v} = \lambda\vec{v}$ ($\vec{v} \neq \vec{0}$): λ is a eigenvalue and $\vec{v}$ is the corresponding eigenvector

• λ is a eigenvalue A if $\det(A - \lambda I_n) = 0$; i.e. λ is a solution of the characteristic equation

• if the eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of A form a basis for $\mathbb{R}^n$ then A is *diagonalizable*: $A = PDP^{-1}$ where D is the diagonal matrix $\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$ and $P = [\vec{v}_1 \,|\, \vec{v}_2 \,|\, \cdots \,|\, \vec{v}_n]$

• if the eigenvalues of A are distinct then there is a basis of eigenvectors

Dot Product

• $\vec{x} \cdot \vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$

• $(\vec{x} + \vec{y}) \cdot \vec{z} = \vec{x} \cdot \vec{z} + \vec{y} \cdot \vec{z}$

• $(c\,\vec{x}) \cdot \vec{y} = c\,(\vec{x} \cdot \vec{y})$

• $\vec{x} \cdot vecx \geq 0$ with equality only if $\vec{x} = 0$

• $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ is the norm of $\vec{x}$.

• $\|c\,\vec{x}\| = c\,\|\vec{x}\|$

• $\|\vec{x} - \vec{y}\|$ = distance from $\vec{x}$ to $\vec{y}$

Orthogonality

• $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an *orthogonal* set of vectors if for all $i \neq j$, $\vec{v}_i \cdot \vec{v}_j = 0$ or more compactly $\vec{v}_i \perp \vec{v}_j$

• $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an *orthonormal* set if it is orthogonal and for each i, $\|\vec{v}_i\| = 1$

• an orthogonal basis for a subspace is a basis which is also orthogonal

• if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an orthogonal basis for a subspace W, then for every $\vec{w}$ in W, $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$ where

$$c_i = \frac{\vec{w} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

• if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an orthogonal basis for a subspace W and $\vec{y}$ is a vector *not* in W then $\hat{y}$, the vector in W *closest* to $\vec{y}$ is $\hat{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$ where

$$c_i = \frac{\vec{y} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

• $\hat{y}$ is the projection of $\vec{y}$ onto W: $\hat{y} = \text{proj}_W(\vec{y})$

• $\|\vec{y} - \hat{y}\| = \text{dist}(\vec{y}, W)$

Gram-Schmidt Procedure

• $W = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$

$$\vdots$$

$$\vec{v}_i = \vec{x}_i - \frac{\vec{x}_i \cdot \vec{v}_{i-1}}{\vec{v}_{i-1} \cdot \vec{v}_{i-1}}\vec{v}_{i-1} - \cdots - \frac{\vec{x}_i \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 - \frac{\vec{x}_i \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$

Least Squares Solutions

• if $A\vec{x} = \vec{b}$ is inconsistent we seek $\widehat{x}$ such that $A\widehat{x}$ is closest to $\vec{b}$; i.e. $\widehat{x}$ minimizes $\|A\vec{x} - \vec{b}\|$; $\widehat{x}$ is the *least squares* solution of $A\vec{x} = \vec{b}$

• a least squares solution to $A\vec{x} = \vec{b}$ is an actual solution to $(A^T A)\,\vec{x} = A^T\vec{b}$

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