1. Determine whether or not the following functions are solutions to the given differential equations.
   
   (a) \( y' = 3x^2, \quad y(x) = x^3 + 5 \)
   
   (b) \( y'' = 9y, \quad y(x) = \sin(3x) \)
   
   (c) \( y'' + 2y = 2y', \quad y(x) = e^{-x} \cos(x) \)
   
   (d) \( y' + 2xy^2 = 0, \quad y(x) = \frac{2}{1 + x^2} \)
   
   (e) \( x^2y'' + xy' - y = \ln(x), \quad y(x) = x - \ln(x) \)
   
   (f) \( y' + 2xy^2 = 0, \quad y(x) = \frac{1}{1 + x^2} \)

2. Find the general solution to the following differential equations.
   
   (a) \( \frac{dy}{dx} = y \sin(x) \)
   
   (b) \( \frac{dy}{dx} + 2xy = 0 \)
   
   (c) \( y^3 \frac{dy}{dx} = (y^4 + 1) \cos(x) \)
   
   (d) \( x^2y' = 1 - x^2 + y^2 - x^2y^2 \)
   
   (e) \( \frac{dy}{dx} = \frac{1 + \sqrt{x}}{1 + \sqrt{y}} \)
   
   (f) \( \frac{dy}{dx} = 4\sqrt{xy} \)
   
   (g) \( y' = -2xy^2 \)

3. Find the solution to the following initial value problems.
   
   (a) \( y' + y = 2, \quad y(0) = 0 \)
   
   (b) \( xy' + y = 3xy, \quad y(1) = 0 \)
   
   (c) \( y' = (1 - y) \cos(x), \quad y(\pi) = 2 \)
(d) $\frac{dy}{dx} - 2y = 3e^{2x}, \quad y(0) = 0$

(e) $\frac{dx}{dt} = 1 + t + x + tx, \quad x(0) = 0$

(f) $y' = 2xy + 3x^2e^{x^2}, \quad y(0) = 5$

(g) $xy' - y = x, \quad y(1) = 7$

4. Determine the eigenvalues of the following matrices.

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} -2 & 5 \\ 4 & -10 \end{pmatrix}$

(d) $\begin{pmatrix} 1/2 & 3/4 \\ 1 & -9/10 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

5. Determine a solution to the following systems of differential equations.

(a)

\[
\begin{align*}
\frac{dx}{dt} &= -x + 3y \\
\frac{dy}{dt} &= -2x + 4y
\end{align*}
\]

(b)

\[
\begin{align*}
\frac{dx}{dt} &= x + 2y \\
\frac{dy}{dt} &= y
\end{align*}
\]

(c) \( \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ -4 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \)

(d) \( \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \)
6. For each of the systems in the previous problem, determine the solution that has the initial condition \( x(0) = -1, y(0) = 4 \).

7. Find constants \( A, B, \ldots \) such that the given function is a solution to the given differential equation.

(a) \( f(t) = A \sin(t) + B \cos(t), \quad \frac{dx}{dt} - 3x = \frac{\cos(t)}{2} \)

(b) \( f(t) = A \cos(\omega t) + B \sin(\omega t), \quad \frac{d^2x}{dt^2} - \frac{3}{4}x = 4 \sin(\omega t) \)

(c) \( f(t) = A + Bt + Ct^2 + Dt^3, \quad \frac{d^3x}{dt^3} - 4t \frac{d^2x}{dt^2} + 3x = t^3 \)

(d) \( f(t) = Ae^{3t} + Bt + C, \quad \frac{d^2x}{dt^2} - \frac{dx}{dt} + 2x = 4e^{3t} - t \)

(e) \( f(t) = A \cos(2t + B), \quad \frac{dx}{dt} - 6x = \cos(2t) \)

8. A tank contains 1000 liters of a solutions consisting of 100kg of salt dissolved in water. Pure water is added into the tank at a rate of 5 L/s and the mixture is pumped out at the same rate. If the solution remains well-mixed, how long will it be until only 10kg of salt remains in the tank?

9. Suppose that a reservoir of 8 million cubic meters initially has a pollutant concentration of 0.25 \%. Suppose further that there is a daily inflow of 500,000 cubic meters of polluted water (with a concentration of 0.05 \%) and an equal outflow of well-mixed water from the reservoir. How long will it take to reduce the pollutant concentration to 0.10 \%? What will be the limiting concentration?

10. A 400 liter tank contains initially 100 L of brine which contains 5 kg of salt. Brine containing 100 g of salt per litre enters at a rate of 5 L / second, and the mixed solution flows out at a rate of 3 L / second. How much salt will be in the tank when it is full of brine?

11. A 3kg mass is attached to a spring with spring coefficient \( k = 48N/m \). The mass is initially 0.5m to the left of equilibrium and at rest when it is let go. If the friction is negligible, find the equation of motion of the spring, together with its amplitude, period, and frequency.

12. A damped mass/spring system is given by

\[
x'' + bx' + 16x = 0
\]
with initial conditions $x(0) = 1, x'(0) = 0$. Find the equation of motion and sketch its graph for $b = 0, 6, 10$.

13. Suppose that a harmonic oscillator with $m = 2, b = 3, k = 1$ is met with an input force of $f(t) = F_0$ for some constant $F_0$. Suppose that the initial conditions are $x(0) = x'(0) = 0$. Find the long-term solution to the spring’s motion. Does this make sense physically?

14. A mass on a spring (with no friction) is acted on by an external force given by $f(t) = F_0 \cos(3\omega t)$. Show that there are two frequencies $\omega$ which will produce resonance. Hint: Try writing $\cos(3\omega t)$ in terms of powers of $\cos(\omega t)$. Alternatively, look up Chebyshev polynomials for some inspiration. Alternatively alternatively, look at Euler’s formula:

$$\cos(nt) + i\sin(nt) = e^{int} = (\cos(t) + i\sin(t))^n.$$  

15. Find the general solution to each of the following ODES.

(a) $x'' - x' - 6x = 2\sin(3t)$
(b) $y''' - y - e^x = 7$
(c) $x'' + 9x = 2t^2e^{3t} + 5$
(d) $y'' - y' - 2y = 3x + 4$
(e) $y''' + 4y' = 3x - 1$
(f) $y'' + 3y' + 2y = xe^{-x} - xe^{-2x}$

16. Write each of the following functions as a power series about the point $t_0 = 0$.

(a) $f(t) = \sin(t^2)(t^2 + 1)$
(b) $g(t) = e^{3t^2} - e^{-3t^2}$
(c) $h(t) = -\int_0^t \ln(u^2)du$

17. For each of the following power series, write them in closed-form.

(a) $\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$
(b) $\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n}}{(2n + 1)!} x^{2n}$
(c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

18. Solve the following initial value problems by substituting in a power series $x(t) = \sum_{n=0}^{\infty} a_n t^n$ and determining the recurrence relation.
(a) \( x' = t^2x, \quad x(0) = 2 \)
(b) \( (t - 1)x' + 2x = 0, \quad x(0) = 1 \)
(c) \( x'' - 2x = 0, \quad x(0) = 0, x'(0) = 1 \)
(d) \( x'' - (1 + t)x' + (1 - t^2)x = 0, \quad x(0) = 2, x'(0) = 0 \)

19. For each of the following differential equations, determine the first four terms of the Frobenius series solutions around the regular singular point \( x_0 = 0 \).

(a) \( 4xy'' + 2y' + y = 0 \)
(b) \( 6x^2y'' + 7xy' - (x^2 + 2)y = 0 \)
(c) \( 2x^2y'' + \sin(x)y' - \cos(x)y = 0 \)

20. The Hermite equation of order \( \alpha \) is given by

\[
y'' - 2xy' + 2\alpha y = 0
\]

Find a power series solution to this for arbitrary \( \alpha \). Furthermore, show that if \( \alpha \) is a positive integer, then at least one of the solutions is in fact a polynomial. Hint: One of the solutions is

\[
y_1(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m 2^m \alpha(\alpha - 2) \cdots (\alpha - 2m + 2)}{(2m)!} x^{2m}
\]

21. Consider the Legendre equation of order \( n \), the differential equation

\[
(1 - t^2)x'' - 2tx' + n(n + 1)x = 0
\]

where \( n \in \mathbb{N} \). Show that the function

\[
P_n(t) = \frac{1}{n!2^n} \frac{d^n}{dt^n} (t^2 - 1)^n
\]

is a solution to this differential equation. Hint: Show first that \( v(t) = (t^2 - 1)^n \) satisfies

\[
(1 - t^2)v'' + 2(n - 1)tv' + 2nv = 0
\]

Now differentiate both sides of this equation \( n \) times and see what you get.

22. Using the above formula, compute the first 4 Legendre polynomials.

23. Consider the differential equation

\[
\frac{d^2F}{dt^2} - \frac{dF}{dt} - F = 0
\]

with initial conditions \( F(0) = F'(0) = 1 \). Assume that \( F(t) \) is of the form

\[
F(t) = \sum_{n=0}^{\infty} \frac{F_n}{n!} t^n
\]

and derive a recurrence relation for the coefficients \( F_n \). Do you recognize them?