Math 232 (Math?)

Intro: What is a differential eq'n?

- Involves derivatives: $\frac{dy}{dx}$

$\Rightarrow$ It is an equation involving derivatives.

Why? We want to build models of phenomena in the world.
We want to see into the future.

We measure rates of change w/ derivatives, and use the resulting "solutions of eqns" to understand future behavior.
\( \frac{dy}{dx} = 0 \)

\( \frac{dy}{dx} + ay = 0 \) where \( a \in \mathbb{R} \)

\( \frac{d^2y}{dx^2} + a \frac{dy}{dx} + b = 0 \) \( a, b \in \mathbb{R} \)

Differential eq'n governs the motion of bodies (e.g. ball thrown in air)

In this case: \( b = \) mass \( a = \) friction

Goal: Come up with a model which describes population growth.

Notation: \( P(t) \) is population as a function of time.
What should this depend on?

We want to write
\[ \frac{dP}{dt} = \ldots ? \]

Things which affect this:
- Rate of death
- Existing population.

It is (hopefully) clear that the larger the population, the faster it grows. \[ \frac{dP}{dt} \] goes up if \( P \) goes up.

Possibilities:
- \[ \frac{dP}{dt} = k_1 P \] \{ increasing functions \}
- \[ \frac{dP}{dt} = k_2 P^2 \] \{ \}

- \[ \frac{dP}{dt} = 73P^9 + 92P^4 + 873422P^{3/2} + P/2 \]
This satisfies our idea, but it is a bit complicated.

Let's start with the simplest:

$$\frac{dP}{dt} = kP$$

This is our first model. (Yay!)

Next: "Solve" this eq'n.

What do we mean by this?

We want to find a function \( P(t) \) which, when substituted into the given eq'n, actually makes it true.

**Guesses:**

a) \( P(t) = 0 \)

**LHS:** \( \frac{dP}{dt} = 0 \)  \hspace{1cm} **RHS:** \( k \cdot 0 = 0 \)
b) $P(t) = t$

Check: LHS: $\frac{dP}{dt} = 1$

RHS: $kP(t) = kt$

2) $P(t) = e^{kt}$

Check: LHS: $\frac{dP}{dt} = k e^{kt}$

RHS: $kP = k(e^{kt})$

Population grows exponentially (*

(If $k < 0$ it decreases!)

Claim: this was apparent from

$\frac{dP}{dt} = kP$

(Note: we are assuming $P > 0$)

If $k < 0$ then RHS < 0

Thus $\frac{dP}{dt} < 0$ and so the population decreases.
If \( k > 0 \), population grows exponentially.

One other thing:
Initial population?

we can find a more general solution:

\[
P(t) = Ce^{kt} \quad (\text{check!})
\]

In such a case: \( P(0) = Ce^{k \cdot 0} = C \)
So \( C \) is the initial population.

ex: Find the solution to the following population models:

\( a) \quad \frac{dP}{dt} = 2P \quad P(0) = 3 \)

\[
\text{[initial value problem]}
\]

\text{Ans.} \quad \text{We see that}

\[
P(t) = Ce^{2t}
\]

is a solution.
Turn: \[ P(0) = C = 3 \]

So
\[ P(t) = 3e^{2t} \]

b) Solve: \[ \frac{dP}{dt} = kP \] where

<table>
<thead>
<tr>
<th>Condition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(0) )</td>
<td>3</td>
</tr>
<tr>
<td>( P(1) )</td>
<td>6</td>
</tr>
</tbody>
</table>

So by
\[ P(t) = Ce^{kt} \]

- \( P(0) = C = 3 \) ---- (1)
- \( P(1) = Ce^k = 6 \) ---- (2)

Thus: (2) becomes
\[ 3e^k = 6 \]

or
\[ e^k = 2 \]

Take \( \ln(\cdot) \) of both sides:
\[ k = \ln 2 \]

Thus:
\[ P(t) = 3e^{t\ln 2} \]
\[ = 3(2^t) \]
Q: What are the flaws of the model?

Read §1.1 up to p. 9
(Limited Resources...)

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