Last week we introduced DEs, we gave a single method to solve them:
\[ \frac{dy}{dx} = f(x)g(y) \]

Then we separate variables...
\[ \int \frac{dy}{g(y)} = \int f(x) \, dx \quad (+ c) \]

This is a great solution... when it works.

But:
\[ \frac{dy}{dx} = y - x \]

is not separable.
Since (as of today) we cannot solve this exactly, what to do?

Pictures!

Ex: Logistic equation

\[
\frac{dP}{dt} = kP(1 - \frac{P}{N})
\]

How to make this picture?
§1.3 Slope Fields

Consider a DE of the form
\[ \frac{dy}{dx} = f(x, y) \]
(i.e. the most general first-order ODE)

Suppose we already knew a solution,
\[ y_1(x) : \]

\[ \text{Slope is } f(x_0, y_0) \]
\[ (x_0, y_0) \rightarrow \text{slope is } f(x_0, y_0) \]

LHS: Slope of \( y_1(x) \)
RHS: Some function \( f(x, y) \)

So the DE tells us what the slope is, along the solution \( y_1(x) \), at every point, which is given by \( f(x, y) \)
This is great, if we know $y'(x)$.
If we don't have the solution...
all we know is $f(x,y)$. i.e.
if the solution were to pass
through $(x_0, y_0)$, then we
would know the slope at
that point.
we can use this!
Ex: Consider $\frac{dy}{dx} = y - x$.
At "every" point in the $(x,y)$-plane,
we draw a "microtangent" with slope
$f(x,y) = y - x$.
This is our slope field. As yet, no clear picture has emerged. However...
"Slope field calculator" on google.

Note: \[
\frac{dy}{dx} = \frac{d}{dx} (x+1) = 1 \quad \text{(LHS)}
\]

\[
y-x = (x+1) - x = 1 \quad \text{(RHS)}
\]

Slope fields let us look at pictures of what solutions will look like. This can often let us understand long-term behavior.
In this case:

what is \( \lim_{x \to \infty} y(x) \)?

Answer: it depends! On what the initial value given!
\[
\frac{dy}{dx} = \frac{(y/8)(4-y)(y-1.5)}{y}
\]

Initial values:
\[y(0) = 2\]

Thus: if \[y(0) = 2\]
then \[\lim_{{t \to \infty}} y(t) = 4\]

Q: What is a good interpretation of the solution \[y(t) = 1.5\] if this is a population model?
Properties of slope fields

1) If the DE is "autonomous"
   i.e. of the form
   \[ \frac{dy}{dx} = f(y) \] (no \* dependence)

Then the slopes don't depend on \( x \)!
   i.e. if we draw them:

```
        y
     /   /
    /     /
   /       /
 /         /
```
   all slopes are the same.

what this tells us: given a solution
   \[ y_1(x) \]
we can find other solutions by translating i.e.
   \[ y_2(x) = y_1(x + \alpha) \]
will always be a solution.

Also: \( \frac{dy}{dx} = f(x) \) \( \rightarrow \) no \( y \) dependence

By FTC: \( y(x) = \int f(x) \, dx + C \) \( \uparrow \) constant shift.

\( \therefore \) the slope field does not depend on \( y \).

i.e. you can always find a new solution by translating up or down.
Equation: $\sin(x)^2 = \frac{dy}{dx}$

$y(0) = 1$

Vertical translates
Equation: $x^2 - y^2 = \frac{dy}{dx}$

So we can always draw approximate solutions.

How can we get more out of this? This lets us "see" solutions and in particular, the long-term behavior, but it does not give us numerical solutions.
one possible method (probably the easiest) comes from these pictures.

"HW": Think about this

Also: Read §1.3 up to "RC circuits"