Last class: What happens when we can't exactly solve an ODE?

1) Slope fields: These make pretty pictures.

What happens if we want a more quantitative description?

Suppose I want \( y(1000) \) given some ODE

\[
\frac{dy}{dt} = f(t, y) \quad y(0) = y_0
\]

(at least, approximately)?

§1.24 Euler's method:

We can approximate solutions with a sequence of straight lines.
How it works: Given

\[ \frac{dy}{dt} = f(t, y) \]

with initial condition \( y(t_0) = y_0 \)

we know that a solution to this would satisfy

\[ \frac{dy}{dt} = f(t_0, y_0) \]

Slope is \( f(t_0, y_0) \)

Pick a "step size" \( \Delta t \).

(t_0, y)

If the Function were a straight line:

\[ y(t_1) \approx y(t_0) + \frac{dy}{dt}(t_0, y_0) \Delta t \]

i.e. \( y \approx y(t_1) \approx y_0 + f(t_0, y_0) \Delta t \).

We can continue:

\[ y_2 = y_1 + f(t_1, y_1) \Delta t \]
Equation: $1 - x^* y$

$y(0) = 1$

$\Delta t = 1$
And so on:

\[ y_{i+1} = y_i + f(t_i, y_i) \Delta t \]

where \( t_i = t_0 + i \Delta t \)

This gives us an iterative approximation to a solution of

\[ \frac{dy}{dt} = f(t, y) \quad y(t_0) = y_0. \]

Example: \( \frac{dy}{dt} = 2y-1 \quad y(0) = 1 \)

\((t_0, y_0) = (0, 1) \quad \Delta t = 0.1\)

Crank the Euler machine:

\[ t_1 = 0 + 0.1 = 0.1 \]

\[ y_1 = y_0 + f(0.1, 1)0.1 = 1 + (2(1)-1)0.1 \]

\[ = 1.1 \]

\[ t_2 = 0.2 \]

\[ y_2 = y_1 + f(0.1, 1.1)0.1 = 1.1 + (2(1.1)-1)0.1 \]

\[ = 1.22 \]
<table>
<thead>
<tr>
<th>$k$</th>
<th>$t_k$</th>
<th>$y_k$</th>
</tr>
</thead>
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</tr>
<tr>
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<td>9</td>
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</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>3.596</td>
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</table>

Thus: $y(1) \approx 3.596$.

So... how close is this?

It turns out: the exact solution to this ODE is

$$y(t) = \frac{e^{2t} - 1}{2}$$

Thus:

$$y(1) = \frac{e^2 - 1}{2} \approx 4.195$$

... not so close.
Why?

Because the solution is concave up, we find that the approx. gets worse and worse.

We would want to choose smaller $\Delta t$ in order to improve this.

However, this increases the no. of steps...

If we chose $\Delta t = 0.05$

\[ y(1) \approx 3.864 \] (better, but not crazy better)

Example 2: \[ \frac{dy}{dt} = -2ty^2 \quad y(0) = 1 \]

we want to estimate \( y(2) \) with $\Delta t = 0.5$
we compute
\[ t_1 = \frac{1}{2}, \quad y_1 = y_0 + f(t_0, y_0) \frac{1}{2} = 1 + 0 \cdot \frac{1}{2} = 1 \]

<table>
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<tr>
<th>k</th>
<th>t_k</th>
<th>y_k</th>
</tr>
</thead>
<tbody>
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<tr>
<td>3</td>
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</table>

What is the actual answer?

Solve by hand:
\[ \frac{dy}{dt} = -2ty^2 \implies \int \frac{dy}{y^2} = \int -2t \, dt + c \]
\[ \frac{1}{y} = t^2 + c \]
\[ y(t) = \frac{1}{t^2 + c} \]
\[ y(0) = \frac{1}{c} = 1 \implies c = 1 \]

Thus the sol'n is \[ y(t) = \frac{1}{t^2 + 1} \]
Comparison to approx:

\[ y(2) = \frac{1}{2^2 + 1} = \frac{1}{5} \]

How close? \[ y(2) - y_4 = \frac{7}{160} \]

If we would have picked a smaller \( \Delta t \) this would have been better!

Note: The given ODE is

\[ \frac{dy}{dt} = e^t \sin(y) \]

This has equilibrium solutions when

\[ y = \pi, 0, 2\pi, 3\pi, \ldots \]

but the slope gets big!

So we need to not just blindly put numbers into computers, but think about what we are doing.
Firstly: No class on Friday!

HW: \[ \begin{align*} 
1.3 & \quad 7, 18, 11, 12, 15, 16, 17 \\
1.4 & \quad 1, 2, 7, 8, 13 
\end{align*} \]