Midterm: HUM AUD
6 - 7:30 pm Thursday
February 13

(But: Strictly speaking, "not mandatory")

ex: Mixing problems: Consider a similar problem to before:

\[ \text{Rate in} \ (r_i) \quad \text{Rate out} \ (r_o) \]

\[ \text{Concentration in} \ (c_i) \quad \text{Concentration out} \ (c_o) \]

Where: \( r_i \) is the rate that fluid flows in
\( c_i \) is the concentration coming in
\( r_o \) is the rate fluid flows out
\( c_o \) is the concentration pours out
Remark: $r_i, r_o$ are in L/min
$c_i, c_o$ are in g/L

(in our case: this is a saline solution
where $c_i, c_o$ are amount of salt/volume
of water)

Let us derive the ODE which describes this
setup:

In a small time interval $\Delta t$, how
much salt $\Delta x$ goes in and out of
our container?

$\Delta x = ?$

Inflow of salt: Should be in grams!

\[
\text{Salt in: } r_i \cdot c_i \cdot \Delta t
\]

Check units: $\left[ \frac{L}{min} \right] \left[ g \right] \left[ min \right] = \left[ g \right]$

\[
\text{Salt out: } r_o \cdot c_o \cdot \Delta t
\]

So: $\Delta x = r_i c_i \Delta t - r_o c_o \Delta t$
So in the interval $\Delta t$, this is a description of the change in the salt amount.

Divide by $\Delta t$:

$$\frac{\Delta x}{\Delta t} = r_i c_i - r_o c_o$$

"let $\Delta t \to 0$"

Yields:

$$\frac{dx}{dt} = r_i c_i - r_o c_o$$

But:

$$c_o = \frac{x}{V}$$

So this is really:

$$\frac{dx}{dt} = r_i c_i - \frac{r_o x}{V}$$

Remark: if $r_i = r_o$ this is much simpler:

$V$ is constant! This is separable and the solution is much easier.

In such a case: $V(t) = V_0 + (r_i - r_o)t$

Thus:

$$\frac{dx}{dt} = r_i c_i - \frac{r_o}{V_0 + (r_i - r_o)t} x$$

Our derivation is complete!
ex: Suppose we are mixing salt in a beaker. We start with 1 L of pure water, and start pouring in saline solution whose concentration is 10 g/L at a rate of 0.1 L/min. We are leaking solution from the beaker at a rate of 0.05 L/min. When will we have 20 g of salt?

Sol'n: We have the ODE which governs the amount of salt:

$$\frac{dx}{dt} = r_1 c_i - \frac{r_o}{V_0 + (r_1 - r_o) t}$$

where: \( r_1 = 0.1 \text{ L/min} \quad c_i = 10 \text{ g/L} \)

\( r_o = 0.05 \text{ L/min} \quad V_0 = 1 \)

$$S_0 \quad \frac{dx}{dt} = (0.1)10 - \frac{0.05}{1 + (0.1 - 0.05)t} \times x$$

$$= 1 - \frac{0.05}{1 + 0.05t}$$

This is of course linear, so we can go and solve it!

Let's use integrating factors:
In this case:
\[ \mu(t) = e^{\int \frac{0.05}{1+0.05t} \, dt} \ln(1+0.05t) \]
\[ = e \]

Thus, we find:
\[ \frac{d}{dt} \left( (1+0.05t)x \right) = 1+0.05t \]
\[ \Rightarrow x(t) = \frac{1}{1+0.05t} \left[ \int (1+0.05t) \, dt + C \right] \]

\[ \int \frac{0.05}{1+0.05t} \, dt \quad du = 0.05 \, dt \]
\[ = \int \frac{1}{u} \, du \quad \text{to compare:} \]
\[ 0.05 \int \frac{1}{1+0.05t} \, dt \quad du = 0.05 \, dt \]

So,
\[ x(t) = \frac{1}{1+0.05t} \left( t + 0.05 \frac{t^2}{2} + C \right) \]
Since \( x(0) = 0 \Rightarrow C = 0 \)
\[ \Rightarrow x(t) = \frac{1}{1+0.05t} \left( t + \frac{t^2}{40} \right) \]
So when is \( x(t) = 20 \)?

\[
20 = \frac{1}{1 + \frac{t}{20}} \left( t + \frac{t^2}{40} \right)
\]

\[
20 + t = t + \frac{t^2}{40} \quad \Rightarrow \quad 800 + 40t = 40t + t^2
\]

or \( t^2 = 800 \quad \Rightarrow \quad t = \pm 10\sqrt{8} \)

\( \approx 20 \sqrt{2} \)

\( \approx 28 \) mins.

Q: If it takes 28 minutes to get 20g of salt, how much water do we have in our beaker?

Review: Linear ODEs:

\[
\frac{dy}{dx} = a(x)y + b(x)
\]

- General solution is \( ay(x) + yp(x) \)
- Methods:
  a) guess!
  b) undetermined coefficients
  c) integrating factors.
3.1.6 Equilibria and the Phase Line

Something we need for next topic:

Given \( \frac{dy}{dt} = f(t, y) \)

we drew slope diagrams.

In the case that this was an autonomous ODE:

\[
\left[ \frac{dy}{dt} = f(y) \right] \quad \text{(no } t \text{ dependence)}
\]

We saw:

a) Solutions are obtained as

\[
\int \frac{dy}{f(y)} = t + C
\]

we had to integrate and invert (i.e. solve for \( y \))

b) Any solution can be obtained by:

\( y_2(t) = y_1(t+C) \) is a soln for "all" \( C \).

This is nice for pictures!
The phase line tells us a lot of info about solutions.

Example:

\[ y(2-y)(1+y) = \frac{dy}{dx} \]

\[ \text{pt is "unstable" } \]

\[ \text{unstable}. \]
Similarly: is "stable"