Recall: Population model for competing species:
\[
\frac{dx}{dt} = 2x(1 - \frac{x}{a}) - xy \\
\frac{dy}{dt} = 3y(1 - \frac{y}{b}) + 2xy
\]

What can we say?

**Equilibria:**
- \((0, 0)\)
- \((2, 0)\)
- \((0, 3)\)
- \((1, 1)\)

With a bit of linear algebra, we can actually get a reasonable visualization.

The key idea: The whole behavior of all solutions is governed by what happens around equilibria!
logistic behavior

Why?
Because linear Algebra!
This is a great application.

We replace the complicated function

\[ f^1(x) = \begin{cases} 
2x(1 - \frac{x}{2}) - xy & \text{if } x < 0 \\
3y(1 - \frac{y}{3}) - 2xy & \text{if } x \geq 0 
\end{cases} \]

with a matrix i.e.

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}
\end{pmatrix}
\]

And this gives us the picture!
Finally, let's draw "solution curves" i.e. let's try include t!
§ 2.3] Damped harmonic Oscillator

Recall: \[ \begin{array}{c}
  m \\
  k
\end{array} \]

\[ \rightarrow \]

Hooke's Law \[ F = -kx \]

Newton's 2nd Law \[ F = m \frac{d^2x}{dt^2} \]

Together: \[ m \frac{d^2x}{dt^2} = -kx \]

But we are neglecting friction!

we want to introduce a **damping** term.

a) This term should be 0 if \( v = 0 \)

b) it should be opposite the direction of motion.

c) As \( v \) increases, so does resistance.

let's pick: \[ -b \cdot v = F_{\text{Friction}} \]
So combining these:

\[ F = -kx - b \frac{dx}{dt} = m \frac{d^2x}{dt^2} \]

How can we solve this?

1. divide by \( m \).

rewrite as

\[ \frac{d^2x}{dt^2} + p \frac{dx}{dt} + q x = 0 \]

where \( p = \frac{b}{m} > 0 \)
\( q = \frac{k}{m} > 0 \)

Example: Consider the hypothetical friction-y spring. with \( p = 5 \) \( q = 6 \)

\[ \text{i.e., } \frac{d^2x}{dt^2} + 5 \frac{dx}{dt} + 6x = 0. \]

First: Split up into: \( v = \frac{dx}{dt} \)

then this is

\[ \frac{dv}{dt} + 5v + 6x = 0 \]
\[ \frac{dx}{dt} - v = 0 \]

We can solve this w/ linear algebra!
Not obvious how to continue.
So...
Let's guess solutions!
(This is a 2nd order linear ODE,
and there is always a canonical guess.)

Since we have
\[ \frac{d^2 x}{dt^2} + 5 \frac{dx}{dt} + 6x = 0 \]
\[ \text{the } P(x) \text{ should look similar to its derivatives.} \]

Guess: \( e^{\lambda t} \) (where we don't know \( \lambda \))

Substitute in:
\[ 0 = \frac{d^2}{dt^2} (e^{\lambda t}) + 5 \frac{dt}{dt} (e^{\lambda t}) + 6 e^{\lambda t} \]
\[ = \lambda^2 e^{\lambda t} + 5 \lambda e^{\lambda t} + 6 e^{\lambda t} = e^{\lambda t} (\lambda^2 + 5 \lambda + 6) \]
\[ \neq 0 \]
\[ = 0 \]
Thus $e^{\lambda t}$ is a solution to this ODE exactly when
\[ \lambda^2 + 5\lambda + 6 = 0 \]

So we should factor this
\[ (\lambda + 2)(\lambda + 3) = 0 \]

Therefore: $\lambda = -2$ and $\lambda = -3$.

Or $e^{-2t}$ and $e^{-3t}$.

are solutions!

(Actually: A generic solution is
\[ x(t) = Ae^{-2t} + Be^{-3t}. \]

(Since the eqn' is linear. See Ch. 3)

Q: What happens as $t \to \infty$?

it goes to 0!
This is not surprising from the perspective of first-order systems:

\[
\frac{dv}{dt} = -sv - 6x \\
\frac{dx}{dt} = v
\]

The equilibria are: The only one is \((0, 0)\).
So if there is a finite limit as \(t \to \infty\), it must be 0.

We can actually sketch solution curves in the \(x-v\) plane:

\begin{align*}
\text{Reason} \\
\text{Linear Algebra!}
\end{align*}