Recall: \[
\frac{dx}{dt} = -2x - 3y
\]
\[
\frac{dy}{dt} = 3x - 2y
\]

\[\mathbf{A} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}\]

The eigenvalues are: \(\lambda_1 = -2 + 3i\)
\(\lambda_2 = -2 - 3i\)

Remark: \(\lambda_1 = \overline{\lambda_2}\) (or \(\lambda_2 = \overline{\lambda_1}\))

We note that, if we have a matrix \(\mathbf{A}\), whose entries are real numbers, then if it has complex eigenvalues they will always be conjugate pairs.

We also computed: \(\mathbf{v}_1 = (1, i)\)
let us compute \( \mathbf{v}_2 \): 

\( \lambda_2 = -2 - 3i \)

i.e. a solution to

\[
\begin{pmatrix}
-2 - \lambda_2 & -3 \\
3 & -2 - \lambda_2
\end{pmatrix}
\begin{pmatrix}
s \\
t
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

i.e.

\[
\begin{pmatrix}
3i & -3 \\
3 & 3i
\end{pmatrix}
\begin{pmatrix}
s \\
t
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

or

\((3i)s - 3t = 0\)

i.e.

\(3i s = 3t\)

\(s = t\)

So if we pick \(s = 1\) we get

\[
\begin{pmatrix}
1 \\
i
\end{pmatrix} = \mathbf{v}_2
\]

Alternatively: pick \(s = -i\) \(\Rightarrow t = 1\)

So

\[
\mathbf{v}_2 = \begin{pmatrix}
-i \\
1
\end{pmatrix}
\]

\[
\begin{pmatrix}
i \\
1
\end{pmatrix} = \bar{\mathbf{v}}_1
\]

Conjugate!

However: if we are modeling a physical system (e.g.\(m\) \[
\begin{array}{c}
m \\
\end{array}
\]

\[
\begin{array}{c}
\text{motion}
\end{array}
\]

\[
\begin{array}{c}
x
\end{array}
\]

\[
\begin{array}{c}
\text{motion}
\end{array}
\]

\[
\begin{array}{c}
x
\end{array}
\]
then we should only want real solutions.

So how to return to reality?

Note: General solution to

\[ \mathbf{\dot{\mathbf{y}}} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \mathbf{\ddot{y}} \]

is

\[ \mathbf{\ddot{y}}(t) = c_1 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(-2+3i)t} + c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(2-3i)t} \]

So to fix:

let us look at \[ \mathbf{\ddot{y}}(t) = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(-2+3i)t} \]

This is:

\[ \mathbf{\ddot{y}}(t) = e^{-2t} \begin{pmatrix} i \\ 1 \end{pmatrix} e^{3it} \]

\[ \cos(3t) + i \sin(3t) \]

\[ = e^{-2t} \begin{pmatrix} i \\ 1 \end{pmatrix} (\cos(3t) + i \sin(3t)) \]
\[ \begin{align*}
&= e^{-2t} \begin{pmatrix}
-\sin(3t) + i \cos(3t) \\
\cos(3t) + i \sin(3t)
\end{pmatrix} \\
&= \begin{pmatrix}
-e^{-2t} \sin(3t) \\
e^{-2t} \cos(3t)
\end{pmatrix} + i \begin{pmatrix}
e^{-2t} \cos(3t) \\
e^{-2t} \sin(3t)
\end{pmatrix} \\
\text{Re}(\tilde{v}(t)) &+ \text{Im}(\tilde{v}(t))
\end{align*} \]

**Theorem:** If \( \tilde{v}(t) \) is a \( \mathbb{C} \)-valued solution to the ODE
\[ \tilde{v}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tilde{v} \]
where \( a, b, c, d \in \mathbb{R} \), then \( \text{Re}(\tilde{v}(t)) \) and \( \text{Im}(\tilde{v}(t)) \) are solutions to the system. In fact, this will span the solution space.
Rephrased: A general solution to

\[
\mathbf{y}' = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \mathbf{y}
\]

will be

\[
\mathbf{y}(t) = c_1 e^{-2t} \begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} \cos(3t) \\ -\sin(3t) \end{pmatrix}
\]

Complex solution

Remark: We only need to look at one eigenvalue and eigenvector. Why?

Important: We can now visualize our solutions!
let's look at the solutions above:

\[ \bar{y}_1 = e^{-2t} \begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix} \]

absent the \( e^{-2t} \), this is just a circle!

Thus, as \( e^{-2t} \to 0 \) as \( t \to \infty \), this spirals inwards:

\[ t = 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

The other solution is similar:

\[ \bar{y}_2(t) = e^{-2t} \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix} \]
which looks like:

Important: always spirals counterclockwise.

Another example: We saw

\[ \frac{d^2 x}{dt^2} + 2 \frac{dx}{dt} + 2 = 0 \]

we noted that solutions were

\[ e^{\lambda t} \quad \text{where} \quad \lambda = -1 \pm i \]

\[ \Rightarrow \text{General solution is} \]

\[ x(t) = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t} \]
Same as before: write
\[ e^{(-1+i)t} = e^{-t} e^{it} \]
\[ = e^{-t} (\cos(t) + i \sin(t)) \]

**General fact:** If
\[ \frac{d^2x}{dt^2} + a \frac{dx}{dt} + bx = 0 \]
where \( a, b \in \mathbb{R} \), and s.t. the eigenvalues are complex, then we obtain all solutions by looking at real & imaginary parts of a single solution.

(important: 2nd order ODE!)

\[ \Rightarrow \] In our case, all solutions are combinations
\[ x(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t) \]

**Remark:** Considering
\[ \frac{d^2x}{dt^2} + x = 0 \]
which has solutions: \( \sin(t) \cos(t) \)
\[ e^{it} \quad e^{-it} \]
Example: determine the solution of
\[
\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2 = 0
\]
\[\text{s.t.: } x(0) = 3 \quad x'(0) = 0\]

we need two conditions to specify a unique solution!

Sol'n: General solution is
\[x(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t)\]

So \[x(0) = c_1 e^0 \cos(0) + c_2 e^0 \sin(0)\]
\[\Rightarrow c_1 = 3\]

Next: Compute \[x'(t) = c_1 (-e^{-t} \cos(t) + e^{-t} \sin(t))\]
\[+ c_2 (-e^{-t} \sin(t) + e^{-t} \cos(t))\]

So \[0 = x'(0) = c_1 + c_2\]
\[\Rightarrow c_2 = -3.\]
Thus our solution is

\[ x(t) = 3e^{-t} \cos(t) - 3e^{-t} \sin(t) \]

\[
\text{[Aside: Redo this w/ the general solution } x(t) = a_1 e^{(-1+i)t} + a_2 e^{(-1-i)t} \text{]}
\]

So what behaviours can occur?
So far we have seen two cases that spiral inward: both \( \dot{x}(t) \) and \( x(t) \), in a phase plane, end up at 0 as \( t \to \infty \).

This is governed by \( \lambda = a + bi \)

We have three cases:

\[ \begin{align*}
\text{.} & \quad a > 0 \\
\text{.} & \quad a = 0 \\
\text{.} & \quad a < 0 \quad \text{— both of our examples.}
\end{align*} \]
In case (1): \((a > 0)\)
The solutions look like
\[ x(t) = c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt) \]
If \(a > 0\) then these terms go to \(\infty\) i.e. it spirals out.

We call this a \(\text{spiral source}\)
(3) if $a < 0$, as we have seen:

This is a spiral sink.

(2) if $a = 0$ then our solutions look like

$$x(t) = c_1 \cos(bt) + c_2 \sin(bt)$$

and so are periodic.
3.1  5, 8, 14, 15, 30

3.2  11, 12, 15, 16

3.3  9, 10

3.4  1, 3, 4, 9, 10, 15

Midterm

In class, Friday
March 21

2×2 matrix