Recall we were considering

$$\frac{d^2x}{dt^2} + 2x = \cos(\omega t)$$

$$x(0) = 0 = x'(0)$$

Case 1) \(\omega^2 \neq 2\)

We saw the solution was

$$x(t) = \frac{1}{2 - \omega^2} \left( \cos(\omega t) - \cos\left(\frac{\sqrt{2}}{2} t\right) \right)$$

particular homogeneous

We saw:

$$x(t) = \frac{1}{2 - \omega^2} \sin\left(\frac{\omega + \sqrt{2}}{2} t\right) \sin\left(\frac{\omega - \sqrt{2}}{2} t\right)$$

"beating" frequencies

Remarks: The amplitude is going to be

$$\frac{1}{|2 - \omega^2|}$$
So as \( \omega \to \sqrt{2} \) we see that the amplitude gets arbitrarily large.

Furthermore: as \( \omega \to \sqrt{2} \) the two periods are

\[
\frac{\omega + \sqrt{2}}{2} \to \sqrt{2} \quad \text{which is the natural period of the harmonic oscillator}
\]

\[
\frac{\omega - \sqrt{2}}{2} \to 0 \quad \text{i.e. we have a longer and longer period!}
\]
So:

If $w$ is not close to $\sqrt{2}$, this may be what we see. There is some interference.

If $w$ is closer to $\sqrt{2}$:
or even more extreme:

where there is a lot of constructive interference.

**Remark:** The general solution is

\[ x(t) = k_1 \cos(\sqrt{2}t) + k_2 \sin(\sqrt{2}t) + \frac{1}{2 - \omega^2} \cos(\omega t) \]

as \( \omega^2 \to 2 \) (or \( \omega \to \sqrt{2} \)) we see that this last term dominates, i.e. the solution looks a lot like the forced solution.
Case 2: \( w = \sqrt{2} \):

We still need a particular solution:

i.e. a solution to

\[
\frac{d^2 x}{dt^2} + 2x = \cos(\sqrt{2}t)
\]

if we picked \( x_p(t) = \alpha \cos(\sqrt{2}t) + \beta \sin(\sqrt{2}t) \)

this doesn't work — it is a homogeneous sol'n.

let's try

\[
x_p(t) = t(\alpha \cos(\sqrt{2}t) + \beta \sin(\sqrt{2}t))
\]

[Remark: This is analogous to when we had forcing exponentials which were homogeneous sol'n's]

So what do \( \alpha, \beta \) have to be?

\[
x''_p(t) = 2(-\alpha \sqrt{2} \cos(\sqrt{2}t) + \beta \sqrt{2} \sin(\sqrt{2}t))
\]

\[
+ t(-2\alpha \cos(\sqrt{2}t) + 2\beta \sin(\sqrt{2}t))
\]
So:

\[
x_p'' + 2x_p = -2\sqrt{2} \alpha \cos(\sqrt{2}t) + 2\sqrt{2} \beta \sin(\sqrt{2}t) + 2t \alpha \cos(\sqrt{2}t) + 2t \beta \sin(\sqrt{2}t)
\]

\[
\Rightarrow x_p(t) = -\frac{1}{2\sqrt{2}} t \cos(\sqrt{2}t)
\]

So the gen'c sol'n is

\[
x(t) = k_1 \cos(\sqrt{2}t) + k_2 \sin(\sqrt{2}t) - \frac{1}{2\sqrt{2}} t \cos(\sqrt{2}t)
\]

if we care about \( x(0) = x'(0) = 0 \)

then our sol'n is

\[
x(t) = -\frac{1}{2\sqrt{2}} t \cos(\sqrt{2}t)
\]
what does a graph of this look like?

Since the "amplitude" is given by $\frac{t}{2\sqrt{2}}$

This is bad for mechanical systems!

The amplitude is unbounded!
Series Solutions:

Consider

\[ 2 \frac{d^2x}{dt^2} + t \frac{dx}{dt} + x = 0 \]

a) This is linear!
b) it is 2nd order
c) non-constant coefficients!

Thus: trying \( x = e^{kt} \) will fail

Try this!

Note: \( x(t) = e^{-t^2/4} \) is a solution.

Check:

\[ x'(t) = (e^{-t^2/4}) \left(-\frac{t}{2}\right) \]

\[ x''(t) = (e^{-t^2/4}) \left(-\frac{t^2}{2}\right) + (e^{-t^2/4}) \left(-\frac{1}{2}\right) \]

So if we combine this:

\[ 2 \frac{t^2}{4} e^{-t^2/4} + \frac{e^{-t^2/4}}{2} \left(-\frac{t^2}{2}\right) e^{-t^2/4} + e^{-t^2/4} = 0 \]
How did we get this?

Power / Taylor series!

i.e. Pretend the solution is of the form

\[ X(t) = \sum_{n=0}^{\infty} a_n t^n \]

\[ = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots \]

We then will want to see what conditions this places on the coefficients.

HW: 4.1 1, 9, 12, 15, 17, 19, 20, 24, 25

4.2 1, 4, 11, 14, 15, 17, 19, 21

4.3 1, 3, 8, 9, 14, 15, 18, 21