Recall: Given an ODE of the form

\[ x'' + p(t)x' + q(t)x = 0 \]

Then we say to is a singular pt. if \( p(t_0) \) or \( q(t_0) \) is singular (i.e. not cts or differentiable).

If, however \( (t - t_0)^2 p(t) \) and \( (t - t_0)^2 q(t) \) are "nice" then this is a regular singular point.

**Example:** \[ x'' + \frac{1 - t}{1 + t} x' + \frac{6 - t^2 + t^20}{(1 + t)^2 q(t)} x = 0 \]

Then \( t_0 = -1 \) is a singular point.

However! \( (1 + t) p(t) = 1 - t \) \( \{ \text{polynomials!} \} \)

\( (1 + t)^2 q(t) = 6 - t^2 + t^{20} \)

This is hence a regular singular point.
Example: Find and classify the singular points of the Bessel eq'n of order n:

\[ x^2 y'' + x y' + (x^2 - n^2) y = 0 \]

Sol'n: Divide by \( x^2 \):

\[ y'' + \frac{1}{x} y' + \frac{x^2 - n^2}{x^2} y = 0 \]

And we see that 0 is a singular point.

Then:

\[ (x - 0) \frac{1}{x} = 0 \]

\[ (x - 0)^2 \frac{x^2 - n^2}{x^2} = x^2 - n^2 \]

\[ \Rightarrow \text{this is a regular singular point.} \]

How to solve?

Consider:

\[ x^2 y'' + \frac{3}{2} x y' - \frac{1}{2} y = 0 \]
Clear: 0 is a regular singular point.

What is a good guess? (Answer)

Try \( y = x^r \)

Then:
\[
\begin{align*}
y' &= r x^{r-1} \\
y'' &= r(r-1) x^{r-2}
\end{align*}
\]

If we substitute in:
\[
\begin{align*}
x^2 \left( r(r-1) x^{r-2} \right) + \left( \frac{3}{2} x \right) r x^{r-1} - \frac{1}{2} x^r \\ &= r(r-1) x^r + \frac{3}{2} r x^r - \frac{1}{2} x^r \\ &= x^r \left[ r(r-1) + \frac{3}{2} r - \frac{1}{2} \right] = 0 \\
&= 0
\end{align*}
\]

This gives us a quadratic equation to determine \( r \)!

we can factor this to find:
\[ r = \frac{1}{2} \quad \text{or} \quad r = -1, \]

i.e.
\[ y_1(x) = x^{\frac{1}{2}} \quad \text{and} \quad y_2(x) = x^{-1} = \frac{1}{x} \]

are two solutions.

So these give us solutions near the singular point \( x_0 = 0 \).

More generally:

if we have
\[ x^2 y'' + p_0 x y' + q_0 x = 0 \]

with \( p_0, q_0 \in \mathbb{R} \).

We guess \( y(x) = x^r \) again:

\[ y' = r x^{r-1} \]
\[ y'' = r (r-1) x^{r-2} \]

Thus:
\[ 0 = r(r-1) x^r + p_0 r x^r + q_0 x^r \]
\[ = x^r \left[ r(r-1) + p_0 r + q_0 \right] \]
So \( x^r \) is a solution if and only if

\[
  r(r-1) + p r + q_0 = 0
\]

**Indicial eq'n**

(we have to consider if there are two roots, etc.)

**Rule:** if \( x^r \) is a solution,
then \( r \) may not be an integer!
it also might be negative!

Suppose now

\[
p(x) = \sum_{n=0}^{\infty} a_n x^n
\]

\[
q(x) = \sum_{n=0}^{\infty} b_n x^n
\]

and consider the ODE

\[
x^2 y'' + p(x)x y' + q(x) y = 0
\]
What should we pick as a guess?

0 is still a regular singular point.

Consider the Frobenius series:

\[ y(x) = \sum_{n=0}^{\infty} c_n x^{n+r} = x^r \sum_{n=0}^{\infty} c_n x^n \]

Remark: This is not in general a power series!

Assume: \( c_0 \neq 0 \). [we can always assume this by choosing \( r \) appropriately]

Then:

\[ y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \]

\[ y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} \]

So if we substitute:
\[ x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left( \sum_{n=0}^{\infty} b_n x^n \right) \left( \sum_{n=0}^{\infty} c_n x^{n+r} \right) \]

\[ = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} \]

Compute the \( x^r \) term: [i.e. the \( n=0 \) terms]

\[ r(r-1)c_0 x^r + a_0 r c_0 x^r + b_0 c_0 x^r \]

\[ = c_0 x^r \left( r^2 + a_0 + b_0 \right) = 0 \]

indicial eq'\( u \)!
So we still get an indicial eq'n which lets us compute \( r \).

\[
\text{[Thus our guess was good!]} \\
\text{[We will always assume the indicial eq'n has two roots]}
\]

**Ex:** Find the \( r \) used in the Frobenius series for the ODE

\[
2x^2 (1 + x) \frac{d^2y}{dx^2} + 3x(1 + x)^3 \frac{dy}{dx} - 6(1 - x^2)y = 0
\]

**Remark:** regular sing. pts @ \( x_0 = 0, -1 \).

But we only care about 0.

Rewrite as:

\[
\frac{3}{2} x^2 y'' + \frac{3}{2} x (1 + x)^2 y' - 3(1 - x)y = 0
\]

\[
p(x) = \frac{3}{2} (1 + x)^2 \quad q(x) = -3(1 - x)
\]

\[
\Rightarrow p_0 = \frac{3}{2} \quad q_0 = -3
\]
So the indicial equation is:

\[ r(r-1) + por + q_0 = 0 \]

\[ r(r-1) + \frac{3}{2} r - 3 = 0 \]

\[ \Rightarrow r = -2, \quad 3/2 \]

Froeb. solutions will be

\[ Y_1(x) = \sum_{n=0}^{\infty} c_n x^{n-2} \quad \text{not power series!} \]

\[ Y_2(x) = \sum_{n=0}^{\infty} c_n x^{n+3/2} \]

How to compute the coefficients \( c_n \)?

We look at higher powers of \( x \) to determine recursions which will compute the other coefficients:
Ex: \[2x^2 y'' + 3xy' - (x^2 + 1)y = 0\]

or: \[x^2 y'' + \left(\frac{3}{2}\right)x y' - \frac{1}{2}(x^2 + 1)y = 0\]

\[p(x) = \frac{3}{2}, \quad q(x) = -\frac{1}{2}(x^2 + 1)\]

\[\Rightarrow r = \frac{1}{2}, -1 \quad \text{(Indicial eq'}'n)\]

look at higher powers of \(x\):

\[\sum_{n=0}^{\infty} \frac{(n+r)(n+r-1)}{2} C_n x^{n+r} + \frac{3}{2} \sum_{n=0}^{\infty} (n+r) C_n x^{n+r} - \frac{1}{2} \sum_{n=0}^{\infty} \alpha_n x^{n+r+2}\]

\[\sum_{n=2}^{\infty} C_{n-2} x^{n+r}\]
look now @ \( n = 1 \):

\[
(1 + r) c_1 x^{r+1} + \frac{3}{2} (1 + r) c_1 x^{r+1} - \frac{1}{2} c_1 x^{r+1} = 0
\]

\[
c_1 x^{r+1} \left[ (1 + r) r - \frac{3}{2} (1 + r) - \frac{1}{2} \right] = 0
\]

Since \( r = -1 \) or \( r = \frac{1}{2} \):

**Case 1** \( r = -1 \):

\[
c_1 x^{-1+1} \left[ 0 - 0 - \frac{1}{2} \right] = c_1 \left( -\frac{1}{2} \right) = 0
\]

\[\Rightarrow c_1 = 0.\]

The \( n = 2 \) term now yields a recursion to determine \( c_2 \) from \( c_0, c_1 \).

This determines the series.