1. The general solution to the associated homogeneous equation is $y_h(t) = ke^{-3t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha e^{-t}$. Then

$$\frac{dy_p}{dt} + 4y_p = -\alpha e^{-t} + 4\alpha e^{-t} = 3\alpha e^{-t}.$$ 

Consequently, we must have $3\alpha = 9$ for $y_p(t)$ to be a solution. Hence, $\alpha = 3$, and the general solution to the nonhomogeneous equation is

$$y(t) = ke^{-3t} + 3e^{-t}.$$

3. The general solution to the associated homogeneous equation is $y_h(t) = ke^{-3t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\frac{dy_p}{dt} + 3y_p = -2\alpha \sin 2t + 2\beta \cos 2t + (3\alpha \cos 2t + 3\beta \sin 2t)$$

$$= (3\alpha + 2\beta) \cos 2t + (3\beta - 2\alpha) \sin 2t$$

Consequently, we must have

$$(3\alpha + 2\beta) \cos 2t + (3\beta - 2\alpha) \sin 2t = 4 \cos 2t$$

for $y_p(t)$ to be a solution. We must solve

$$\begin{cases} 
3\alpha + 2\beta = 4 \\
3\beta - 2\alpha = 0.
\end{cases}$$

Hence, $\alpha = 12/13$ and $\beta = 8/13$. The general solution is

$$y(t) = ke^{-3t} + \frac{12}{13} \cos 2t + \frac{8}{13} \sin 2t.$$
4. The general solution to the associated homogeneous equation is \( y_h(t) = ke^{2t} \). For a particular solution of the nonhomogeneous equation, we guess \( y_p(t) = \alpha \cos 2t + \beta \sin 2t \). Then
\[
\frac{dy_p}{dt} - 2y_p = -2\alpha \sin 2t + 2\beta \cos 2t - 2(\alpha \cos 2t + \beta \sin 2t)
= (2\beta - 2\alpha) \cos 2t + (-2\alpha - 2\beta) \sin 2t.
\]
Consequently, we must have
\[
(2\beta - 2\alpha) \cos 2t + (-2\alpha - 2\beta) \sin 2t = \sin 2t
\]
for \( y_p(t) \) to be a solution, that is, we must solve
\[
\begin{align*}
-2\alpha - 2\beta &= 1 \\
-2\alpha + 2\beta &= 0.
\end{align*}
\]
Hence, \( \alpha = -1/4 \) and \( \beta = -1/4 \). The general solution of the nonhomogeneous equation is
\[
y(t) = ke^{2t} - \frac{1}{4} \cos 2t - \frac{1}{4} \sin 2t.
\]

6. The general solution of the associated homogeneous equation is \( y_h(t) = ke^{t/2} \). For a particular solution of the nonhomogeneous equation, we guess \( y_p(t) = \alpha te^{t/2} \) rather than \( \alpha e^{t/2} \) because \( \alpha e^{t/2} \) is a solution of the homogeneous equation. Then
\[
\frac{dy_p}{dt} - \frac{y_p}{2} = \alpha e^{t/2} + \frac{\alpha}{2} te^{t/2} - \frac{\alpha te^{t/2}}{2}
= \alpha e^{t/2}.
\]
Consequently, we must have \( \alpha = 4 \) for \( y_p(t) \) to be a solution. Hence, the general solution to the nonhomogeneous equation is
\[
y(t) = ke^{t/2} + 4te^{t/2}.
\]
7. The general solution to the associated homogeneous equation is \( y_h(t) = ke^{-2t} \). For a particular solution of the nonhomogeneous equation, we guess a solution of the form \( y_p(t) = \alpha e^{t/3} \). Then

\[
\frac{dy_p}{dt} + 2y_p = \frac{1}{3} \alpha e^{t/3} + 2\alpha e^{t/3}
\]

\[
= \frac{7}{3} \alpha e^{t/3}.
\]

Consequently, we must have \( \frac{7}{3} \alpha = 1 \) for \( y_p(t) \) to be a solution. Hence, \( \alpha = 3/7 \), and the general solution to the nonhomogeneous equation is

\[ y(t) = ke^{-2t} + \frac{3}{7} e^{t/3}. \]

Since \( y(0) = 1 \), we have

\[ 1 = k + \frac{3}{7}, \]

so \( k = 4/7 \). The function \( y(t) = \frac{4}{7} e^{-2t} + \frac{3}{7} e^{t/3} \) is the solution of the initial-value problem.

10. The general solution of the associated homogeneous equation is \( y_h(t) = ke^{-3t} \). For a particular solution of the nonhomogeneous equation, we guess a solution of the form \( y_p(t) = \alpha \cos 2t + \beta \sin 2t \). Then

\[
\frac{dy_p}{dt} + 3y_p = -2\alpha \sin 2t + 2\beta \cos 2t + 3\alpha \cos 2t + 3\beta \sin 2t
\]

\[
= (3\alpha + 2\beta) \cos 2t + (-2\alpha + 3\beta) \sin 2t.
\]

Consequently, we must have

\[
(3\alpha + 2\beta) \cos 2t + (-2\alpha + 3\beta) \sin 2t = \cos 2t
\]

for \( y_p(t) \) to be a solution. We must solve

\[
\begin{align*}
3\alpha + 2\beta &= 1 \\
-2\alpha + 3\beta &= 0.
\end{align*}
\]

Hence, \( \alpha = 3/13 \) and \( \beta = 2/13 \). The general solution to the differential equation is

\[ y(t) = ke^{-3t} + \frac{3}{13} \cos 2t + \frac{2}{13} \sin 2t. \]

To find the solution of the given initial-value problem, we evaluate the general solution at \( t = 0 \) and obtain

\[ y(0) = k + \frac{3}{13}. \]

Since the initial condition is \( y(0) = -1 \), we see that \( k = -16/13 \). The desired solution is

\[ y(t) = -\frac{16}{13} e^{-3t} + \frac{3}{13} \cos 2t + \frac{2}{13} \sin 2t. \]
12. The general solution to the associated homogeneous equation is $y_h(t) = ke^{2t}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha te^{2t}$ rather than $\alpha e^{2t}$ because $\alpha e^{2t}$ is a solution of the homogeneous equation. Then

$$\frac{dy_p}{dt} - 2y_p = \alpha e^{2t} + 2\alpha ie^{2t} - 2\alpha ie^{2t}$$

$$= \alpha e^{2t}.$$

Consequently, we must have $\alpha = 7$ for $y_p(t)$ to be a solution. Hence, the general solution to the nonhomogeneous equation is

$$y(t) = ke^{2t} + 7te^{2t}.$$

Note that $y(0) = k$, so the solution to the initial-value problem is

$$y(t) = 3e^{2t} + 7te^{2t} = (7t + 3)e^{2t}.$$
21. To find the general solution, we use the technique suggested in Exercise 19. We calculate two particular solutions—one for the right-hand side \( r^2 + 2r + 1 \) and one for the right-hand side \( e^{4t} \).

With the right-hand side \( r^2 + 2r + 1 \), we guess a solution of the form 
\[
y_{p_1}(t) = at^2 + bt + c.
\]

Then 
\[
\frac{d}{dt}y_{p_1} + 2y_{p_1} = 2at + b + 2(at^2 + bt + c)
= 2at^2 + (2a + 2b)t + (b + 2c).
\]

Then \( y_{p_1} \) is a solution if
\[
\begin{cases}
2a = 1 \\
2a + 2b = 2 \\
b + 2c = 1.
\end{cases}
\]

We get \( a = 1/2, b = 1/2, \) and \( c = 1/4 \).

With the right-hand side \( e^{4t} \), we guess a solution of the form 
\[
y_{p_2}(t) = \alpha e^{4t}.
\]

Then 
\[
\frac{d}{dt}y_{p_2} + 2y_{p_2} = 4\alpha e^{4t} + 2\alpha e^{4t} = 6\alpha e^{4t},
\]
and \( y_{p_2} \) is a solution if \( \alpha = 1/6 \).

The general solution of the associated homogeneous equation is \( y_h(t) = ke^{-2t} \), so the general solution of the original equation is 
\[
ke^{-2t} + \frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{8} + \frac{1}{6}e^{4t}.
\]

To find the solution that satisfies the initial condition \( y(0) = 0 \), we evaluate the general solution at \( t = 0 \) and obtain 
\[
k + \frac{1}{4} + \frac{1}{8} = 0.
\]

Hence, \( k = -5/12 \).
24. To find the general solution, we use the technique suggested in Exercise 19. We calculate two particular solutions—one for the right-hand side \( \cos 2t + 3 \sin 2t \) and one for the right-hand side \( e^{-t} \).

With the right-hand side \( \cos 2t + 3 \sin 2t \), we guess a solution of the form

\[
y_{p_1}(t) = \alpha \cos 2t + \beta \sin 2t.
\]

Then

\[
\frac{dy_{p_1}}{dt} + y_{p_1} = -2\alpha \sin 2t + 2\beta \cos 2t + \alpha \cos 2t + \beta \sin 2t
\]

\[
= (\alpha + 2\beta) \cos 2t + (-2\alpha + \beta) \sin 2t.
\]

Then \( y_{p_1} \) is a solution if

\[
\begin{cases}
\alpha + 2\beta = 1 \\
-2\alpha + \beta = 3.
\end{cases}
\]

We get \( \alpha = -1 \) and \( \beta = 1 \).

With the right-hand side \( e^{-t} \), making a guess of the form \( y_{p_2}(t) = ae^{-t} \) does not lead to a solution of the nonhomogeneous equation because the general solution of the associated homogeneous equation is \( y_h(t) = ke^{-t} \).

Consequently, we guess

\[
y_{p_2}(t) = ate^{-t}.
\]

Then

\[
\frac{dy_{p_2}}{dt} + y_{p_2} = a(1 - t)e^{-t} + ate^{-t} = ae^{-t},
\]

and \( y_{p_2} \) is a solution if \( a = 1 \).

The general solution of the original equation is

\[
ke^{-t} - \cos 2t + \sin 2t + te^{-t}.
\]

To find the solution that satisfies the initial condition \( y(0) = 0 \), we evaluate the general solution at \( t = 0 \) and obtain

\[
k - 1 = 0.
\]

Hence, \( k = 1 \).
1. We rewrite the equation in the form
\[ \frac{dy}{dt} + \frac{y}{t} = 2 \]

and note that the integrating factor is
\[ \mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t. \]

Multiplying both sides by \( \mu(t) \), we obtain
\[ t \frac{dy}{dt} + y = 2t. \]

Applying the Product Rule to the left-hand side, we see that this equation is the same as
\[ \frac{d(ty)}{dt} = 2t, \]

and integrating both sides with respect to \( t \), we obtain
\[ ty = t^2 + c, \]

where \( c \) is an arbitrary constant. The general solution is
\[ y(t) = \frac{1}{t} (t^2 + c) = t + \frac{c}{t}. \]
3. We rewrite the equation in the form
\[ \frac{dy}{dt} + \frac{y}{1+t} = t^2 \]

and note that the integrating factor is
\[ \mu(t) = e^{\int \frac{1}{1+t} \, dt} = e^{\ln(1+t)} = 1 + t. \]

Multiplying both sides by \( \mu(t) \), we obtain
\[ (1 + t) \frac{dy}{dt} + y = (1 + t)t^2. \]

Applying the Product Rule to the left-hand side, we see that this equation is the same as
\[ \frac{d((1 + t)y)}{dt} = t^3 + t^2, \]

and integrating both sides with respect to \( t \), we obtain
\[ (1 + t)y = \frac{t^4}{4} + \frac{t^3}{3} + c, \]

where \( c \) is an arbitrary constant. The general solution is
\[ y(t) = \frac{3t^4 + 4t^3 + 12c}{12(t + 1)}. \]

6. Note that the integrating factor is
\[ \mu(t) = e^{\int (-2/t) \, dt} = e^{-2 \ln t} = e^{\ln(t^{-2})} = t^{-2}. \]

Multiplying both sides by \( \mu(t) \), we obtain
\[ t^{-2} \frac{dy}{dt} - 2t^{-3} y = te^t. \]

Applying the Product Rule to the left-hand side, we see that this equation is the same as
\[ \frac{d(t^{-2}y)}{dt} = te^t, \]

and integrating both sides with respect to \( t \), we obtain
\[ t^{-2}y = (t - 1)e^t + c, \]

where \( c \) is an arbitrary constant. The general solution is
\[ y(t) = t^2(t - 1)e^t + ct^2. \]
8. We rewrite the equation in the form
\[
\frac{dy}{dt} - \frac{1}{t + 1} y = 4t^2 + 4t
\]
and note that the integrating factor is
\[
\mu(t) = e^{\int \frac{-1}{(t+1)} dt} = e^{-\ln(t+1)} = \left( e^{\ln((t+1)^{-1})} \right) = \frac{1}{t + 1}.
\]
Multiplying both sides by \( \mu(t) \), we obtain
\[
\frac{1}{t + 1} \frac{dy}{dt} - \frac{1}{(t + 1)^2} y = \frac{4t^2 + 4t}{t + 1}.
\]
Applying the Product Rule to the left-hand side, we see that this equation is the same as
\[
\frac{d}{dt} \left( \frac{y}{t + 1} \right) = 4t.
\]
Integrating both sides with respect to \( t \), we obtain
\[
\frac{y}{t + 1} = 2t^2 + c,
\]
where \( c \) is an arbitrary constant. The general solution is
\[
y(t) = (2t^2 + c)(t + 1) = 2t^3 + 2t^2 + ct + c.
\]
To find the solution that satisfies the initial condition \( y(1) = 10 \), we evaluate the general solution at \( t = 1 \) and obtain \( c = 3 \). The desired solution is
\[
y(t) = 2t^3 + 2t^2 + 3t + 3.
\]

9. In Exercise 1, we derived the general solution
\[
y(t) = t + \frac{c}{t}.
\]
To find the solution that satisfies the initial condition \( y(1) = 3 \), we evaluate the general solution at \( t = 1 \) and obtain \( c = 2 \). The desired solution is
\[
y(t) = t + \frac{2}{t}.
\]
11. Note that the integrating factor is

\[ \mu(t) = e^{\int -(2/t) \, dt} = e^{-2 \int (1/t) \, dt} = e^{-2 \ln t} = e^{\ln(t^{-2})} = \frac{1}{t^2}. \]

Multiplying both sides by \( \mu(t) \), we obtain

\[ \frac{1}{t^2} \frac{dy}{dt} - \frac{2y}{t^3} = 2. \]

Applying the Product Rule to the left-hand side, we see that this equation is the same as

\[ \frac{d}{dt} \left( \frac{y}{t^2} \right) = 2, \]

and integrating both sides with respect to \( t \), we obtain

\[ \frac{y}{t^2} = 2t + c, \]

where \( c \) is an arbitrary constant. The general solution is

\[ y(t) = 2t^3 + ct^2. \]

To find the solution that satisfies the initial condition \( y(-2) = 4 \), we evaluate the general solution at \( t = -2 \) and obtain

\[ -16 + 4c = 4. \]

Hence, \( c = 5 \), and the desired solution is

\[ y(t) = 2t^3 + 5t^2. \]
19. We rewrite the equation in the form
\[ \frac{dy}{dt} - ay = 4e^{-t^2} \]
and note that the integrating factor is
\[ \mu(t) = e^{\int (-at) \, dt} = e^{-at^2/2}. \]

Multiplying both sides by \( \mu(t) \), we obtain
\[ e^{-at^2/2} \frac{dy}{dt} - a e^{-at^2/2} y = 4e^{-t^2} e^{-at^2/2}. \]

Applying the Product Rule to the left-hand side and simplifying the right-hand side, we see that this equation is the same as
\[ \frac{d(e^{-at^2/2}y)}{dt} = 4e^{-(1+a/2)t^2}. \]

Integrating both sides with respect to \( t \), we obtain
\[ e^{-at^2/2} y = \int 4e^{-(1+a/2)t^2} \, dt. \]

The integral on the right-hand side can be expressed in terms of elementary functions only if \( 1 + a/2 = 0 \) (that is, if the factor involving \( e^{t^2} \) really isn’t there). Hence, the only value of \( a \) that yields an integral we can express in terms of elementary functions form is \( a = -2 \) (see Exercise 4).
23. The integrating factor is

\[ \mu(t) = e^{\int 2dt} = e^{2t}. \]

Multiplying both sides by \( \mu(t) \), we obtain

\[ e^{2t} \frac{dy}{dt} + 2e^{2t}y = 3e^{2t}e^{-2t} \]

\[ = 3. \]

Applying the Product Rule to the left-hand side, we see that this equation is the same as

\[ \frac{d(e^{2t}y)}{dt} = 3, \]

and integrating both sides with respect to \( t \), we obtain

\[ e^{2t}y = 3t + k, \]

where \( k \) is an arbitrary constant. The general solution is

\[ y(t) = (3t + k)e^{-2t}. \]

We know that \( ke^{-2t} \) is the general solution of the associated homogeneous equation, so \( y_p(t) = 3te^{-2t} \) is a particular solution of the nonhomogeneous equation. Note that the factor of \( t \) arose after we multiplied the right-hand side of the equation by the integrating factor and ended up with the constant 3. After integrating, the constant produces a factor of \( t \).