1. (a) $\mathbf{V}(x, y) = (1, 0)$
   (e) As $t$ increases, solutions move along horizontal lines toward the right.

2. (a) $\mathbf{V}(x, y) = (x, 1)$
   (e) As $t$ increases, solutions move up and right if $x(0) > 0$, up and left if $x(0) < 0$.

3. (a) $\mathbf{V}(y, v) = (-v, y)$
   (e) As $t$ increases, solutions move on circles around $(0, 0)$ in the counter-clockwise direction.
8. (a) Let $v = dy/dt$. Then
\[
\frac{dv}{dt} = \frac{d^2y}{dt^2} = -2y.
\]
Thus the associated vector field is
\[V(y, v) = (v, -2y)\].

(b) See part (c).

(c) As $t$ increases, solutions move around the origin on ovals in the clockwise direction.

9. (a)

(b) The solution tends to the origin along the line $y = -x$ in the $xy$-phase plane. Therefore both $x(t)$ and $y(t)$ tend to zero as $t \to \infty$.

11. (a) There are equilibrium points at $(\pm 1, 0)$, so only systems (ii) and (vii) are possible. Since the direction field points toward the $x$-axis if $y \neq 0$, the equation $dy/dt = y$ does not match this field. Therefore, system (vii) is the system that generated this direction field.

(b) The origin is the only equilibrium point, so the possible systems are (iii), (iv), (v), and (viii). The direction field is not tangent to the $y$-axis, so it does not match either system (iv) or (v). Vectors point toward the origin on the line $y = x$, so $dy/dt = dx/dt$ if $y = x$. This condition is not satisfied by system (iii). Consequently, this direction field corresponds to system (viii).

(c) The origin is the only equilibrium point, so the possible systems are (iii), (iv), (v), and (viii). Vectors point directly away from the origin on the $y$-axis, so this direction field does not correspond to systems (iii) and (viii). Along the line $y = x$, the vectors are more vertical than horizontal. Therefore, this direction field corresponds to system (v) rather than system (iv).

(d) The only equilibrium point is $(1, 0)$, so the direction field must correspond to system (vi).
13. (a) To find the equilibrium points, we solve the system of equations
\[
\begin{align*}
4x - 7y + 2 & = 0 \\
3x + 6y - 1 & = 0.
\end{align*}
\]
These simultaneous equations have one solution, \((x, y) = \left(-\frac{1}{9}, \frac{2}{9}\right)\).

(b) 
\[
\begin{array}{c}
\text{(c) As } t \text{ increases, typical solutions spiral away from the origin in the counter-clockwise direction.}
\end{array}
\]
17. (a) To find the equilibrium points, we solve the system of equations

\[
\begin{align*}
  y &= 0 \\
-\cos x - y &= 0.
\end{align*}
\]

We see that \( y = 0 \), and thus \( \cos x = 0 \). The equilibrium points are \((\pi/2 + k\pi, 0)\) for any integer \(k\).

(b) ![Diagrams showing vector fields and trajectories]

(c) As \( t \) increases, typical solutions spiral toward one of the equilibria on the \( x \)-axis. Which equilibrium point the solution approaches depends on the initial condition.
19. (a) Let \( v = \frac{dx}{dt} \). Then

\[
\frac{dv}{dt} = \frac{d^2x}{dt^2} = 3x - x^3 - 2v.
\]

Thus the associated vector field is \( \mathbf{V}(x, v) = (v, 3x - x^3 - 2v) \).

(b) Setting \( \mathbf{V}(x, v) = (0, 0) \) and solving for \((x, v)\), we get \( v = 0 \) and \( 3x - x^3 = 0 \). Hence, the equilibria are \((x, v) = (0, 0)\) and \((x, v) = (\pm\sqrt{3}, 0)\).

(c)

(d)

(e) As \( t \) increases, almost all solutions spiral to one of the two equilibria \((\pm\sqrt{3}, 0)\). There is a curve of initial conditions that divides these two phenomena. It consists of those initial conditions for which the corresponding solutions tend to the equilibrium point at \((0, 0)\).
21. (a) The \( x(t) \)- and \( y(t) \)-graphs are periodic, so they correspond to a solution curve that returns to its initial condition in the phase plane. In other words, its solution curve is a closed curve. Since the amplitude of the oscillation of \( x(t) \) is relatively large, these graphs must correspond to the outermost closed solution curve.

(b) The graphs are not periodic, so they cannot correspond to the two closed solution curves in the phase portrait. Both graphs cross the \( t \)-axis. The value of \( x(t) \) is initially negative, then becomes positive and reaches a maximum, and finally becomes negative again. Therefore, the corresponding solution curve is the one that starts in the second quadrant, then travels through the first and fourth quadrants, and finally enters the third quadrant.

(c) The graphs are not periodic, so they cannot correspond to the two closed solution curves in the phase portrait. Only one graph crosses the \( t \)-axis. The other graph remains negative for all time. Note that the two graphs cross.

The corresponding solution curve is the one that starts in the second quadrant and crosses the \( x \)-axis and the line \( y = x \) as it moves through the third quadrant.

(d) The \( x(t) \)- and \( y(t) \)-graphs are periodic, so they correspond to a solution curve that returns to its initial condition in the phase plane. In other words, its solution curve is a closed curve. Since the amplitude of the oscillation of \( x(t) \) is relatively small, these graphs must correspond to the innermost closed solution curve.
1. (a) See part (c).
(b) We guess that there are solutions of the form \( y(t) = e^{st} \) for some choice of the constant \( s \). To determine these values of \( s \), we substitute \( y(t) = e^{st} \) into the left-hand side of the differential equation, obtaining
\[
\frac{d^2 y}{dt^2} + 7\frac{dy}{dt} + 10y = \frac{d^2(e^{st})}{dt^2} + 7\frac{d(e^{st})}{dt} + 10(e^{st})
\]
\[
= s^2 e^{st} + 7se^{st} + 10e^{st}
\]
\[
= (s^2 + 7s + 10)e^{st}
\]
In order for \( y(t) = e^{st} \) to be a solution, this expression must be 0 for all \( t \). In other words,
\[
s^2 + 7s + 10 = 0.
\]
This equation is satisfied only if \( s = -2 \) or \( s = -5 \). We obtain two solutions, \( y_1(t) = e^{-2t} \) and \( y_2(t) = e^{-5t} \), of this equation.
(c)
2. (a) See part (c).

(b) We guess that there are solutions of the form \( y(t) = e^{st} \) for some choice of the constant \( s \). To determine these values of \( s \), we substitute \( y(t) = e^{st} \) into the left-hand side of the differential equation, obtaining

\[
\frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = \frac{d^2(e^{st})}{dt^2} + 5 \frac{d(e^{st})}{dt} + 6(e^{st})
\]

\[
= s^2 e^{st} + 5se^{st} + 6e^{st}
\]

\[
= (s^2 + 5s + 6)e^{st}
\]

In order for \( y(t) = e^{st} \) to be a solution, this expression must be 0 for all \( t \). In other words,

\[
s^2 + 5s + 6 = 0.
\]

This equation is satisfied only if \( s = -3 \) or \( s = -2 \). We obtain two solutions, \( y_1(t) = e^{-3t} \) and \( y_2(t) = e^{-2t} \), of this equation.

(c)
5.  (a) See part (c).

(b) We guess that there are solutions of the form \( y(t) = e^{st} \) for some choice of the constant \( s \). To determine these values of \( s \), we substitute \( y(t) = e^{st} \) into the left-hand side of the differential equation, obtaining

\[
\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} - 10y = \frac{d^2(e^{st})}{dt^2} + 3 \frac{d(e^{st})}{dt} - 10(e^{st})
\]

\[
= s^2 e^{st} + 3s e^{st} - 10e^{st}
\]

\[
= (s^2 + 3s - 10)e^{st}
\]

In order for \( y(t) = e^{st} \) to be a solution, this expression must be 0 for all \( t \). In other words,

\[
s^2 + 3s - 10 = 0.
\]

This equation is satisfied only if \( s = -5 \) or \( s = 2 \). We obtain two solutions, \( y_1(t) = e^{-5t} \) and \( y_2(t) = e^{2t} \), of this equation.

(c)
7. \(a\) Let \(y_p(t)\) be any solution of the damped harmonic oscillator equation and \(y_g(t) = \alpha y_p(t)\) where \(\alpha\) is a constant. We substitute \(y_g(t)\) into the left-hand side of the damped harmonic oscillator equation, obtaining

\[
m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = m \frac{d^2 y_p}{dt^2} + b \frac{dy_p}{dt} + ky_p
\]

\[
= m \alpha \frac{d^2 y_p}{dt^2} + b \alpha \frac{dy_p}{dt} + \alpha ky_p
\]

\[
= \alpha \left( m \frac{d^2 y_p}{dt^2} + b \frac{dy_p}{dt} + ky_p \right)
\]

Since \(y_p(t)\) is a solution, we know that the expression in the parentheses is zero. Therefore, \(y_g(t) = \alpha y_p(t)\) is a solution of the damped harmonic oscillator equation.

\(b\) Substituting \(y(t) = \alpha e^{-t}\) into the left-hand side of the damped harmonic oscillator equation, we obtain

\[
\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = \frac{d^2 (\alpha e^{-t})}{dt^2} + 3 \frac{d(\alpha e^{-t})}{dt} + 2(\alpha e^{-t})
\]

\[
= \alpha e^{-t} - 3 \alpha e^{-t} + 2 \alpha e^{-t}
\]

\[
= (\alpha - 3 \alpha + 2 \alpha) e^{-t}
\]

\[
= 0.
\]

We also get zero if we substitute \(y(t) = \alpha e^{-2t}\) into the equation.

\(c\) If we obtain one nonzero solution to the equation with the guess-and-test method, then we obtain an infinite number of solutions because there are infinitely many constants \(\alpha\).
8. (a) Let $y_1(t)$ and $y_2(t)$ be any two solutions of the damped harmonic oscillator equation. We substitute $y_1(t) + y_2(t)$ into the left-hand side of the equation, obtaining

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = m \frac{d^2(y_1 + y_2)}{dt^2} + b \frac{d(y_1 + y_2)}{dt} + k(y_1 + y_2)$$

$$= \left( m \frac{d^2y_1}{dt^2} + b \frac{dy_1}{dt} + ky_1 \right) + \left( m \frac{d^2y_2}{dt^2} + b \frac{dy_2}{dt} + ky_2 \right)$$

$$= 0 + 0 = 0$$

because $y_1(t)$ and $y_2(t)$ are solutions.

(b) In the section, we saw that $y_1(t) = e^{-t}$ and $y_2(t) = e^{-2t}$ are two solutions to this differential equation. Note that the $y_1(0) + y_2(0) = 2$ and $v_1(0) + v_2(0) = -3$. Consequently, $y(t) = y_1(t) + y_2(t)$, that is, $y(t) = e^{-t} + e^{-2t}$, is the solution of the initial-value problem.

(c) If we combine the result of part (a) of Exercise 7 with the result in part (a) of this exercise, we see that any function of the form

$$y(t) = \alpha e^{-t} + \beta e^{-2t}$$

is a solution if $\alpha$ and $\beta$ are constants. Evaluating $y(t)$ and $v(t) = y'(t)$ at $t = 0$ yields the two equations

$$\alpha + \beta = 3$$

$$-\alpha - 2\beta = -5.$$  

We obtain $\alpha = 1$ and $\beta = 2$. The desired solution is $y(t) = e^{-t} + 2e^{-2t}$.

(d) Given that any constant multiple of a solution yields another solution and that the sum of any two solutions yields another solution, we see that all functions of the form

$$y(t) = \alpha e^{-t} + \beta e^{-2t}$$

where $\alpha$ and $\beta$ are constants are solutions. Therefore, we obtain an infinite number of solutions to this equation.
9. We choose the left wall to be the position \( x = 0 \) with \( x > 0 \) indicating positions to the right. Each spring exerts a force on the mass. If the position of the mass is \( x \), then the left spring is stretched by the amount \( x - L_1 \). Therefore, the force \( F_1 \) exerted by this spring is
\[
F_1 = k_1 (L_1 - x) .
\]

Similarly, the right spring is stretched by the amount \( (1 - x) - L_2 \). However, the restoring force \( F_2 \) of the right spring acts in the direction of increasing values of \( x \). Therefore, we have
\[
F_2 = k_2 ((1 - x) - L_2) .
\]

Using Newton’s second law, we have
\[
m \frac{d^2x}{dt^2} = k_1 (L_1 - x) + k_2 ((1 - x) - L_2) - b \frac{dx}{dt},
\]
where the term involving \( dx/dt \) represents the force due to damping. After a little algebra, we obtain
\[
m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + (k_1 + k_2)x = k_1 L_1 - k_2 L_2 + k_2.\]