1. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is
\[ s^2 - s - 6, \]
so the eigenvalues are \( s = -2 \) and \( s = 3 \). Hence, the general solution of the homogeneous equation is
\[ k_1 e^{-2t} + k_2 e^{3t}. \]

To find a particular solution of the forced equation, we guess \( y_p(t) = ke^{4t} \). Substituting into the left-hand side of the differential equation gives
\[
\frac{d^2y_p}{dt^2} - \frac{dy_p}{dt} - 6y_p = 16ke^{4t} - 4ke^{4t} - 6ke^{4t} = 6ke^{4t}.
\]

In order for \( y_p(t) \) to be a solution of the forced equation, we must take \( k = 1/6 \). The general solution of the forced equation is
\[ y(t) = k_1 e^{-2t} + k_2 e^{3t} + \frac{1}{6} e^{4t}. \]

9. First we derive the general solution. The characteristic polynomial is
\[ s^2 + 6s + 8, \]
so the eigenvalues are \( s = -2 \) and \( s = -4 \). To find the general solution of the forced equation, we also need a particular solution. We guess \( y_p(t) = ke^{-t} \) and find that \( y_p(t) \) is a solution only if \( k = 1/3 \). Therefore, the general solution is
\[ y(t) = k_1 e^{-2t} + k_2 e^{-4t} + \frac{1}{3} e^{-t}. \]

To find the solution with the initial conditions \( y(0) = y'(0) = 0 \), we compute
\[ y'(t) = -2k_1 e^{-2t} - 4k_2 e^{-4t} - \frac{1}{3} e^{-t}. \]

Then we evaluate at \( t = 0 \) and obtain the simultaneous equations
\[
\begin{align*}
    k_1 + k_2 + \frac{1}{3} &= 0 \\
    -2k_1 - 4k_2 - \frac{1}{3} &= 0.
\end{align*}
\]

Solving, we have \( k_1 = -1/2 \) and \( k_2 = 1/6 \), so the solution of the initial-value problem is
\[ y(t) = -\frac{1}{2} e^{-2t} + \frac{1}{6} e^{-4t} + \frac{1}{3} e^{-t}. \]
12. This is the same equation as Exercise 6. The general solution is
\[ y(t) = k_1 e^{-2t} + k_2 e^{-5t} + \frac{1}{3} t e^{-2t}. \]

To find the solution with the initial conditions \( y(0) = y'(0) = 0 \), we compute
\[ y'(t) = -2k_1 e^{-2t} - 5k_2 e^{-5t} + \frac{1}{3} e^{-2t} - \frac{2}{3} t e^{-2t}. \]

Then we evaluate at \( t = 0 \) and obtain the simultaneous equations
\[
\begin{aligned}
    k_1 + k_2 &= 0 \\
    -2k_1 - 5k_2 + \frac{1}{3} &= 0.
\end{aligned}
\]

Solving, we have \( k_1 = -1/9 \) and \( k_2 = 1/9 \), so the solution of the initial-value problem is
\[ y(t) = -\frac{1}{9} e^{-2t} + \frac{1}{9} e^{-5t} + \frac{1}{3} t e^{-2t}. \]
15. (a) The characteristic polynomial of the unforced equation is
\[ s^2 + 4s + 3. \]
So the eigenvalues are \( s = -1 \) and \( s = -3 \), and the general solution of the unforced equation is
\[ k_1 e^{-t} + k_2 e^{-3t}. \]
To find a particular solution of the forced equation, we guess \( y_p(t) = ke^{-4t} \). Substituting \( y_p(t) \) into the left-hand side of the differential equation gives
\[ \frac{d^2 y_p}{dt^2} + 4 \frac{dy_p}{dt} + 3y_p = 16ke^{-4t} - 16ke^{-4t} + 3ke^{-4t} = 3ke^{-4t}. \]
So \( k = 1/3 \) yields a solution of the forced equation.

The general solution of the forced equation is therefore
\[ y(t) = k_1 e^{-t} + k_2 e^{-3t} + \frac{1}{3} e^{-4t}. \]

(b) The derivative of the general solution is
\[ y'(t) = -k_1 e^{-t} - 3k_2 e^{-3t} - \frac{4}{3} e^{-4t}. \]
To find the solution with \( y(0) = y'(0) = 0 \), we evaluate at \( t = 0 \) and obtain the simultaneous equations
\[
\begin{align*}
    k_1 + k_2 + \frac{1}{3} &= 0 \\
    -k_1 - 3k_2 - \frac{4}{3} &= 0.
\end{align*}
\]
Solving, we find that \( k_1 = 1/6 \) and \( k_2 = -1/2 \), so the solution of the initial-value problem is
\[ y(t) = \frac{1}{6} e^{-t} - \frac{1}{2} e^{-3t} + \frac{1}{3} e^{-4t}. \]

(c) In the general solution, all three terms tend to zero, so the solution tends to zero. We can say a little more by noting that the term \( k_1 e^{-t} \) is much larger (provided \( k_1 \neq 0 \)). Hence, most solutions tend to zero at the rate of \( e^{-t} \). If \( k_1 = 0 \), then solutions tend to zero at the rate of \( e^{-3t} \) provided \( k_2 \neq 0 \).
17. (a) The characteristic polynomial of the unforced equation is

\[ s^2 + 4s + 20. \]

So the eigenvalues are \( s = -2 \pm 4i \), and the general solution of the unforced equation is

\[ k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t. \]

To find a particular solution of the forced equation, we guess \( y_p(t) = ke^{-2t} \). Substituting \( y_p(t) \) into the left-hand side of the differential equation gives

\[
\frac{d^2 y_p}{dt^2} + 4 \frac{dy_p}{dt} + 20y_p = 4ke^{-2t} - 8ke^{-2t} + 20ke^{-2t} \\
= 16ke^{-2t}.
\]

So \( k = 1/16 \) yields a solution of the forced equation.

The general solution of the forced equation is therefore

\[ y(t) = k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t + \frac{1}{16} e^{-2t}. \]

(b) The derivative of the general solution is

\[ y'(t) = -2k_1 e^{-2t} \cos 4t - 4k_1 e^{-2t} \sin 4t - 2k_2 e^{-2t} \sin 4t + 4k_2 e^{-2t} \cos 4t - \frac{1}{8} e^{-2t}. \]

To find the solution with \( y(0) = y'(0) = 0 \), we evaluate at \( t = 0 \) and obtain the simultaneous equations

\[
\begin{align*}
 k_1 + \frac{1}{16} &= 0 \\
 -2k_1 + 4k_2 - \frac{1}{8} &= 0.
\end{align*}
\]

Solving, we find that \( k_1 = -1/16 \) and \( k_2 = 0 \), so the solution of the initial-value problem is

\[ y(t) = -\frac{1}{16} e^{-2t} \cos 4t + \frac{1}{16} e^{-2t}. \]

(c) Every solution tends to zero like \( e^{-2t} \) and all but one exponential solution oscillates with frequency \( 2/\pi \).
19. The natural guesses of \( y_p(t) = ke^{-t} \) and \( y_p(t) = kte^{-t} \) fail to be solutions of the forced equation because they are both solutions of the unforced equation. (The characteristic polynomial of the unforced equation is 
\[ s^2 + 2s + 1, \]
which has \(-1\) as a double root.)

So we guess \( y_p(t) = kt^2e^{-t} \). Substituting this guess into the left-hand side of the differential equation gives
\[
\frac{d^2y_p}{dt^2} + 2\frac{dy_p}{dt} + y_p = (2ke^{-t} - 4kte^{-t} + kt^2e^{-t}) + 2(2kte^{-t} - kt^2e^{-t}) + kt^2e^{-t} = 2ke^{-t}.
\]

So \( k = 1/2 \) yields the solution
\[
y_p(t) = \frac{1}{2}t^2e^{-t}.
\]

From the characteristic polynomial, we know that the general solution of the unforced equation is 
\[ k_1e^{-t} + k_2te^{-t}. \]
Consequently, the general solution of the forced equation is 
\[ y(t) = k_1e^{-t} + k_2te^{-t} + \frac{1}{2}t^2e^{-t}. \]

20. If we guess a constant function of the form \( y_p(t) = k \), then substituting \( y_p(t) \) into the left-hand side of the differential equation yields
\[
\frac{d^2(k)}{dt^2} + p\frac{d(k)}{dt} + qk = 0 + 0 + qk = qk.
\]

Since the right-hand side of the differential equation is simply the constant \( c \), \( k = c/q \) yields a constant solution.
24. (a) The characteristic polynomial of the unforced equation is

\[ s^2 + 4s + 6. \]

So the eigenvalues are \( s = -2 \pm i\sqrt{2} \), and the general solution of the unforced equation is

\[ k_1 e^{-2t} \cos \sqrt{2} t + k_2 e^{-2t} \sin \sqrt{2} t. \]

To find one solution of the forced equation, we guess the constant function \( y_p(t) = k \). Substituting \( y_p(t) \) into the left-hand side of the differential equation, we obtain

\[ \frac{d^2 y_p}{dt^2} + 4 \frac{dy_p}{dt} + 6y_p = 0 + 4 \cdot 0 + 6k = 6k. \]

Hence, \( k = -4/3 \) yields a solution of the forced equation. The general solution of the forced equation is

\[ y(t) = k_1 e^{-2t} \cos \sqrt{2} t + k_2 e^{-2t} \sin \sqrt{2} t - \frac{4}{3}. \]

(b) To find the solution satisfying the initial conditions \( y(0) = y'(0) = 0 \), we compute the derivative of the general solution

\[ y'(t) = -2k_1 e^{-2t} \cos \sqrt{2} t - \sqrt{2} k_1 e^{-2t} \sin \sqrt{2} t \\
-2k_2 e^{-2t} \sin \sqrt{2} t + \sqrt{2} k_2 e^{-2t} \cos \sqrt{2} t. \]

Using the initial conditions and evaluating \( y(t) \) and \( y'(t) \) at \( t = 0 \), we obtain the simultaneous equations

\[
\begin{aligned}
    k_1 - \frac{4}{3} &= 0 \\
-2k_1 + \sqrt{2} k_2 &= 0.
\end{aligned}
\]

Solving for \( k_1 \) and \( k_2 \) gives \( k_1 = 4/3 \) and \( k_2 = 4\sqrt{2}/3 \). The solution of the initial-value problem is

\[ y(t) = \frac{4}{3} e^{-2t} \cos \sqrt{2} t - \frac{4\sqrt{2}}{3} e^{-2t} \sin \sqrt{2} t - \frac{4}{3}. \]
25. (a) The characteristic polynomial of the unforced equation is

\[ s^2 + 9. \]

So the eigenvalues are \( s = \pm 3i \), and the general solution of the unforced equation is

\[ k_1 \cos 3t + k_2 \sin 3t. \]

To find one solution of the forced equation, we guess \( y_p(t) = ke^{-t} \). Substituting \( y_p(t) \) into the left-hand side of the differential equation, we obtain

\[
\frac{d^2y_p}{dt^2} + 9y_p = ke^{-t} + 9ke^{-t} = 10ke^{-t}.
\]

Hence, \( k = 1/10 \) yields a solution of the forced equation. The general solution of the forced equation is

\[ y(t) = k_1 \cos 3t + k_2 \sin 3t + \frac{1}{10}e^{-t}. \]

(b) To find the solution satisfying the initial conditions \( y(0) = y'(0) = 0 \), we compute the derivative of the general solution

\[ y'(t) = -3k_1 \sin 3t + 3k_2 \cos 3t - \frac{1}{10}e^{-t}. \]

Using the initial conditions and evaluating \( y(t) \) and \( y'(t) \) at \( t = 0 \), we obtain the simultaneous equations

\[
\begin{aligned}
    k_1 + \frac{1}{10} &= 0 \\
    3k_2 - \frac{1}{10} &= 0.
\end{aligned}
\]

Solving for \( k_1 \) and \( k_2 \) gives \( k_1 = -1/10 \) and \( k_2 = 1/30 \). The solution of the initial-value problem is

\[ y(t) = -\frac{1}{10} \cos 3t + \frac{1}{30} \sin 3t + \frac{1}{10}e^{-t}. \]

(c) Since the function \( e^{-t}/10 \to 0 \) quickly, the solution quickly approaches a solution of the unforced oscillator.
1. Recalling that the real part of \( e^{it} \) is \( \cos t \), we see that the complex version of this equation is
\[
\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = e^{it}.
\]

To find a particular solution, we guess \( y_c(t) = ae^{it} \). Then \( dy_c/dt = iae^{it} \) and \( d^2 y_c/dt^2 = -ae^{it} \). Substituting these derivatives into the equation and collecting terms yields
\[
(-a + 3ia + 2a)e^{-it} = e^{it},
\]
which is satisfied if
\[
(1 + 3i)a = 1.
\]

Hence, we must have
\[
a = \frac{1}{1 + 3i} = \frac{1}{10} - \frac{3}{10}i.
\]

So
\[
y_c(t) = \frac{1 - 3i}{10} e^{it} = \frac{1 - 3i}{10} (\cos t + i \sin t)
\]
is a particular solution of the complex version of the equation. Taking the real part, we obtain the solution
\[
y(t) = \frac{1}{10} \cos t + \frac{3}{10} \sin t.
\]

To produce the general solution of the homogeneous equation, we note that the characteristic polynomial \( s^2 + 3s + 2 \) has roots \( s = -2 \) and \( s = -1 \). So the general solution is
\[
y(t) = k_1 e^{-2t} + k_2 e^{-t} + \frac{1}{10} \cos t + \frac{3}{10} \sin t.
\]

4. This equation is the same as the equation in Exercise 3 except for the coefficient of 2 on the right-hand side. The complex version of the equation is
\[
\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = 2e^{it},
\]
and the guess of the particular solution \( y_c(t) = ae^{it} \) yields
\[
a = \frac{1 - 3i}{5}
\]
via the same steps as in Exercise 3. Taking the imaginary part of
\[
y_c(t) = \frac{1 - 3i}{5} e^{it} = \frac{1 - 3i}{5} (\cos t + i \sin t)
\]
and adding the general solution of the homogeneous equation yields
\[
y(t) = k_1 e^{-2t} + k_2 e^{-t} + \frac{1}{2} \sin t - \frac{3}{2} \cos t.
\]
11. From Exercise 5, we know that the general solution of this equation is
\[ y(t) = k_1 e^{-4t} + k_2 e^{-2t} + \frac{7}{85} \cos t + \frac{6}{85} \sin t. \]

To find the desired solution, we must solve for \(k_1\) and \(k_2\) using the initial conditions. We have
\[
\begin{cases}
    k_1 + k_2 + \frac{7}{85} = 0 \\
    -4k_1 - 2k_2 + \frac{6}{85} = 0.
\end{cases}
\]

We obtain \(k_1 = 2/17\) and \(k_2 = -1/5\). The desired solution is
\[ y(t) = \frac{2}{17} e^{-4t} - \frac{1}{5} e^{-2t} + \frac{7}{85} \cos t + \frac{6}{85} \sin t. \]

14. First we find the general solution of the differential equation using the Extended Linearity Principle and the standard guess-and-test technique. The complex version of the equation is
\[ \frac{d^2}{dt^2} + 2 \frac{dy}{dt} + y = 2e^{2it}, \]

and we guess \(y_c(t) = ae^{2it}\) as a particular solution. Substituting this guess into the equation yields
\[ a = \frac{2}{-3 + 4i} = \frac{-6 - 8i}{25}. \]

Hence, a particular solution is the real part of
\[ y_c(t) = \frac{-6 - 8i}{25} \cos 2t + i \sin 2t. \]

We have
\[ y(t) = \frac{-6}{25} \cos 2t + \frac{8}{25} \sin 2t. \]

To find the general solution of the homogeneous equation, we note that the characteristic polynomial is \(s^2 + 2s + 1\), which has \(s = -1\) as a double root. Hence, the general solution of the original equation is
\[ y(t) = k_1 e^{-t} + k_2 te^{-t} - \frac{6}{25} \cos 2t + \frac{8}{25} \sin 2t. \]

To obtain the desired initial conditions, we solve for \(k_1\) and \(k_2\) using
\[
\begin{cases}
    k_1 - \frac{6}{25} = 0 \\
    -k_1 + k_2 + \frac{16}{25} = 0.
\end{cases}
\]

We see that \(k_1 = 6/25\) and \(k_2 = -2/5\), so the desired solution is
\[ y(t) = \frac{6}{25} e^{-t} - \frac{2}{5} te^{-t} - \frac{6}{25} \cos 2t + \frac{8}{25} \sin 2t. \]
15. (a) If we guess
\[ y_p(t) = a \cos 3t + b \sin 3t, \]
then
\[ y'_p(t) = -3a \sin 3t + 3b \cos 3t \]
and
\[ y''_p(t) = -9a \cos 3t - 9b \sin 3t. \]
Substituting this guess and its derivatives into the differential equation gives
\[ (-8a + 9b) \cos 3t + (-9a - 8b) \sin 3t = \cos 3t. \]
Thus \( y_p(t) \) is a solution if \( a \) and \( b \) satisfy the simultaneous equations
\[
\begin{cases}
-8a + 9b = 1 \\
-9a - 8b = 0.
\end{cases}
\]
Solving these equations for \( a \) and \( b \), we obtain \( a = -8/145 \) and \( b = 9/145 \), so
\[ y_p(t) = -\frac{8}{145} \cos 3t + \frac{9}{145} \sin 3t \]
is a solution.

(b) If we guess
\[ y_p(t) = A \cos(3t + \phi), \]
then
\[ y'_p(t) = -3A \sin(3t + \phi) \]
and
\[ y''_p(t) = -9A \cos(3t + \phi). \]
Substituting this guess and its derivatives into the differential equation yields
\[ -8A \cos(3t + \phi) - 9A \sin(3t + \phi) = \cos 3t. \]
Using the trigonometric identities for the sine and cosine of the sum of two angles, we have
\[ -8A (\cos 3t \cos \phi - \sin 3t \sin \phi) - 9A (\sin 3t \cos \phi + \cos 3t \sin \phi) = \cos 3t. \]
This equation can be rewritten as
\[ (-8A \cos \phi - 9A \sin \phi) \cos 3t + (8A \sin \phi - 9A \cos \phi) \sin 3t = \cos 3t. \]
It holds if
\[
\begin{cases}
-8A \cos \phi - 9A \sin \phi = 1 \\
9A \cos \phi - 8A \sin \phi = 0.
\end{cases}
\]
Multiplying the first equation by 9 and the second by 8 and adding yields
\[ 145A \sin \phi = -9. \]
Similarly, multiplying the first equation by $-8$ and the second by $9$ and adding yields

$$145A \cos \phi = -8.$$ 

Taking the ratio gives

$$\frac{\sin \phi}{\cos \phi} = \tan \phi = \frac{9}{8}.$$ 

Also, squaring both $145A \sin \phi = -9$ and $145A \cos \phi = -8$ yields

$$145^2 A^2 \cos^2 \phi + 145^2 A^2 \sin^2 \phi = 145,$$

so $A^2 = 1/145$.

We can use either $A = 1/\sqrt{145}$ or $A = -1/\sqrt{145}$, but this choice of sign for $A$ affects the value of $\phi$. If we pick $A = -1/\sqrt{145}$, then $\sqrt{145} \sin \phi = 9$, $\sqrt{145} \cos \phi = 8$, and $\tan \phi = 9/8$. In this case, $\phi = \arctan(9/8)$. Hence, a particular solution of the original equation is

$$y_p(t) = \frac{1}{\sqrt{145}} \cos \left(3t + \arctan \frac{9}{8}\right).$$
17. Since $p$ and $q$ are both positive, the solution of the homogeneous equation (the unforced response) tends to zero. Hence, we can match solutions to equations by considering the period (or frequency) and the amplitude of the steady-state solution (forced response). We also need to consider the rate at which solutions tend to the steady-state solution.

(a) The steady-state solution has period $2\pi/3$, and since the period of the steady-state solution is the same as the period of the forcing function, these solutions correspond to equations (v) or (vi). Moreover, this observation applies to the solutions in part (d) as well. Therefore, we need to match equations (v) and (vi) with the solutions in parts (a) and (d).

Solutions approach the steady-state faster in (d) than in (a). To distinguish (v) from (vi), we consider their characteristic polynomials. The characteristic polynomial for (v) is

$$s^2 + 5s + 1,$$

which has eigenvalues $(-5 \pm \sqrt{21})/2$. The characteristic polynomial for (vi) is

$$s^2 + s + 1,$$

which has eigenvalues $(-1 \pm i\sqrt{3})/2$. The rate of approach to the steady-state for (v) is determined by the slow eigenvalue $(-5 + \sqrt{21})/2 \approx -0.21$. The rate of approach to the steady-state for (vi) is determined by the real part of the eigenvalue, $-0.5$. Therefore, the graphs in part (a) come from equation (v), and the graphs in part (d) come from equation (vi).

(b) The steady-state solution has period $2\pi$, and since the period of the steady-state solution is the same as the period of the forcing function, these solutions correspond to equations (i) or (ii). Moreover, this observation applies to the solutions in part (c) as well. Therefore, we need to match equations (i) and (ii) with the solutions in parts (b) and (c).

The amplitude of the steady-state solution is larger in (b) than in (c). To distinguish (i) from (ii), we calculate the amplitudes of the steady-state solutions for (i) and (ii). If we complexify these equations, we get

$$\frac{d^2y}{dt^2} + p \frac{dy}{dt} + qy = e^{it}.$$

Guessing a solution of the form $\gamma_c(t) = ae^{it}$, we see that

$$a = \frac{1}{(q - 1) + pi}.$$

The amplitude of the steady-state solution is $|a|$. For equation (i), $|a| = 1/\sqrt{29} \approx 0.19$, and for equation (ii), it is $1/\sqrt{5} \approx 0.44$. Therefore, the graphs in part (b) correspond to equation (ii), and the graphs in part (c) correspond to equation (i).

(c) See the answer to part (b).

(d) See the answer to part (a).
19. (a) Using the fact that the real part of $e^{(-1+i)t}$ is $e^{-t} \cos t$, the complex version of this equation is

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = e^{(-1+i)t}.$$ 

Guessing $y_c(t) = ae^{(-1+i)t}$ yields

$$a(-1 + i)^2e^{(-1+i)t} + 4a(-1 + i)e^{(-1+i)t} + 20ae^{(-1+i)t} = e^{(-1+i)t}.$$ 

Simplifying we have

$$a(16 + 2i)e^{(-1+i)t} = e^{(-1+i)t}.$$ 

Thus, $y_c(t)$ is a solution of the complex differential equation if $a = 1/(16 + 2i)$, and we have

$$y_c(t) = \left(\frac{4}{65} - \frac{1}{130}i\right)e^{-t}(\cos t + i \sin t).$$

So one solution of the original equation is

$$y_p(t) = \frac{4}{65}e^{-t}\cos t + \frac{1}{130}e^{-t}\sin t.$$ 

To find the general solution of the homogeneous equation, we note that the characteristic polynomial $s^2 + 4s + 20$ has roots $s = -2 \pm 4i$.

Hence, the general solution of the original equation is

$$y(t) = k_1e^{-2t}\cos 4t + k_2e^{-2t}\sin 4t + \frac{4}{65}e^{-t}\cos t + \frac{1}{130}e^{-t}\sin t.$$ 

(b) All four terms in the general solution tend to zero as $t \to \infty$. Hence, all solutions tend to zero as $t \to \infty$. The terms with factors of $e^{-2t}$ tend to zero very quickly, which leaves the terms of the particular solution $y_p(t)$ as the largest terms, so all solutions are asymptotic to $y_p(t)$. Since the solution $y_p(t)$ oscillates with period $2\pi$ and the amplitude of its oscillations decreases at the rate of $e^{-t}$, all solutions oscillate with this period and decaying amplitude.

21. Note that the real part of

$$(a - bi)(\cos \omega t + i \sin \omega t)$$

is $g(t)$. Hence, we must find $k$ and $\phi$ such that

$$ke^{i\phi} = a - bi.$$ 

Using the polar form of the complex number $z = a - bi$, we see that

$$ke^{i\phi} = a - bi = |z|e^{i\theta},$$

where $\theta$ is the polar angle for $z$ (see Appendix C). Therefore, we can choose

$$k = |z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \phi = \theta.$$
1. The complex version of this equation is

\[ \frac{d^2y}{dt^2} + 9y = e^{it}. \]

Guessing \( y_c(t) = ae^{it} \) as a particular solution and substituting this guess into the left-hand side of the differential equation yields

\[ 8ae^{it} = e^{it}. \]

Thus, \( y_c(t) \) is a solution if \( 8a = 1 \). The real part of

\[ y_c(t) = \frac{1}{8} e^{it} = \frac{1}{8}(\cos t + i \sin t) \]

is \( y(t) = \frac{1}{8} \cos t \). This \( y(t) \) is a solution to the original differential equation. [Because there is no \( dy/dt \)-term (no damping), we could have guessed a solution of the form \( y(t) = a \cos t \) instead of using the complex version of the equation.]

To find the general solution of the homogeneous equation, we note that the characteristic polynomial is \( s^2 + 9 \), which has roots \( s = \pm 3i \). So the general solution of the original equation is

\[ y(t) = k_1 \cos 3t + k_2 \sin 3t + \frac{1}{8} \cos t. \]

3. The complex version of this equation is

\[ \frac{d^2y}{dt^2} + 4y = -e^{it/2}. \]

Guessing \( y_c(t) = ae^{it/2} \) as a particular solution and substituting into the equation yields

\[ \frac{15}{4} ae^{it/2} = -e^{it/2}. \]

Thus, \( y_c(t) \) is a solution if \( \frac{15}{4} a = -1 \). The real part of

\[ y_c(t) = -\frac{4}{15} e^{it/2} = -\frac{4}{15} \left( \cos \frac{t}{2} + i \sin \frac{t}{2} \right) \]

is

\[ y(t) = -\frac{4}{15} \cos \frac{t}{2}. \]

This \( y(t) \) is a solution to the original differential equation. [Because there is no \( dy/dt \)-term (no damping), we could have guessed a solution of the form \( y(t) = a \cos t/2 \) instead of using the complex version of the equation.]

To find the general solution of the homogeneous equation, we note that the characteristic polynomial is \( s^2 + 4 \), which has roots \( s = \pm 2i \). So the general solution of the original equation is

\[ y(t) = k_1 \cos 2t + k_2 \sin 2t - \frac{4}{15} \cos \frac{t}{2}. \]
8. The complex version of this equation is

\[ \frac{d^2y}{dt^2} + 5y = 5e^{5it}. \]

Guessing \( y_c(t) = ae^{5it} \) as a particular solution and substituting this guess into the left-hand side of the differential equation yields

\[ -20ae^{5it} = 5e^{5it}. \]

Thus, \( y_c(t) \) is a solution if \(-20a = 5\). The imaginary part of

\[ y_c(t) = -\frac{1}{4}(\cos 5t + i \sin 5t) \]

is \( y(t) = -\frac{1}{4} \sin 5t \). This \( y(t) \) is a solution to the original differential equation. [Because there is no \( dy/dt \)-term (no damping), we could have guessed a solution of the form \( y(t) = a \sin 5t \) instead of using the complex version of the equation.]

To find the general solution of the homogeneous equation, we note that the characteristic polynomial is \( s^2 + 5 \), which has roots \( s = \pm \sqrt{5}i \).

Hence, the general solution of the original problem is

\[ y(t) = k_1 \cos \sqrt{5}t + k_2 \sin \sqrt{5}t - \frac{1}{4} \sin 5t. \]

9. From Exercise 1, we know that the general solution is

\[ y(t) = k_1 \cos 3t + k_2 \sin 3t + \frac{1}{8} \cos t. \]

So

\[ y'(t) = -3k_1 \sin 3t + 3k_2 \cos 3t - \frac{1}{8} \sin t. \]

Using the initial conditions \( y(0) = 0 \) and \( y'(0) = 0 \), we obtain the simultaneous equations

\[
\begin{align*}
k_1 + \frac{1}{8} &= 0 \\
3k_2 &= 0,
\end{align*}
\]

which imply that \( k_1 = -1/8 \) and \( k_2 = 0 \). The solution to the initial-value problem is

\[ y(t) = -\frac{1}{8} \cos 3t + \frac{1}{8} \cos t. \]
14. First we find the general solution by considering the complex version of the equation
\[
\frac{d^2y}{dt^2} + 4y = e^{3it}.
\]

Guessing a particular solution of the form \(y_c(t) = ae^{3it}\) and substituting this guess into the left-hand side of the equation yields
\[
-5ae^{3it} = e^{3it}.
\]

Thus, \(y_c(t)\) is a solution if \(a = -1/5\). The imaginary part of
\[
y_c(t) = -\frac{1}{5}(\cos 3t + i \sin 3t)
\]
is \(y(t) = -\frac{1}{5} \sin 3t\). This \(y(t)\) is a solution to the original differential equation. [Because there is no \(dy/dt\)-term (no damping), we could have guessed a solution of the form \(y(t) = a \sin 3t\) instead of using the complex version of the equation.]

To find the general solution of the homogeneous equation, we note that the characteristic polynomial is \(s^2 + 4\), which has roots \(s = \pm 2i\). Hence, the general solution is
\[
y(t) = k_1 \cos 2t + k_2 \sin 2t - \frac{1}{5} \sin 3t.
\]

Note that
\[
y'(t) = -2k_1 \sin 2t + 2k_2 \cos 2t - \frac{3}{5} \cos 3t.
\]

From the initial condition \(y(0) = 2\), we have \(k_1 = 2\). Using the initial condition \(y'(0) = 0\), we obtain the equation \(2k_2 - \frac{3}{5} = 0\), which implies that \(k_2 = \frac{3}{10}\). The solution to the initial-value problem is
\[
y(t) = 2 \cos 2t + \frac{3}{10} \sin 2t - \frac{1}{5} \sin 3t.
\]
15. The characteristic polynomial of the unforced equation is \( s^2 + 4 \), which has roots \( s = \pm 2i \). So the natural frequency is \( 2/(2\pi) \), and the forcing frequency is \( 9/(8\pi) \).

(a) The frequency of the beats is

\[
\frac{9/4 - 2}{4\pi} = \frac{1}{16\pi},
\]

and therefore, the period of one beat is \( 16\pi \approx 50 \).

(b) The frequency of the rapid oscillations is

\[
\frac{9/4 + 2}{4\pi} = \frac{17}{16\pi}.
\]

Therefore, there are 17 rapid oscillations in each beat.

(c)

18. Note that the characteristic polynomial of the homogeneous equation is \( s^2 + 6 \), which has roots \( s = \pm i\sqrt{6} \). So the natural frequency is \( \sqrt{6}/(2\pi) \), and the forcing frequency is \( 2/(2\pi) \).

(a) The frequency of the beats is

\[
\frac{\sqrt{6} - 2}{4\pi},
\]

and therefore, the period of one beat is approximately 28.

(b) The frequency of the rapid oscillations is

\[
\frac{\sqrt{6} + 2}{4\pi}.
\]

Therefore, there are approximately 10 rapid oscillations in each beat.

(c)
21. (a) The graph shows either the solution of a resonant equation or one with beats whose period is very large. The period of the beats in equation (iii) is $4\pi$, and the period of the beats in equation (iv) is $4\pi/(4 - \sqrt{14}) \approx 48.6$. Hence this graph must correspond to a solution of the resonant equation—equation (v).

(b) The graph has beats with period $4\pi$. Therefore, this graph corresponds to equation (iii).

(c) This solution has no beats and no change in amplitude. Therefore, it corresponds to either (i), (ii), or (vi). Note that the general solution of equation (i) is

$$k_1 \cos 4t + k_2 \sin 4t + \frac{5}{8},$$

and the general solution of equation (ii) is

$$k_1 \cos 4t + k_2 \sin 4t - \frac{5}{8}.$$

Equation (iv) has a steady-state solution whose oscillations are centered about $y = 0$. Since the oscillations shown are centered around a positive constant, this function is a solution of equation (i).

(d) The graph has beats with a period that is approximately 50. Therefore, this graph corresponds to equation (iv) (see part (a)).