1a). Find the unique solution to the following initial value problem
\[ \frac{dy}{dt} = \frac{t^2}{y}, \quad y(1) = 0 \]

**Solution:**
This is a separable equation, what we need to do is to separate the variables, and then integrate:
\[ y \, dy = t^2 \, dt \implies \int y \, dy = \int t^2 \, dt \]
\[ \implies \frac{1}{2} y^2 = \frac{2}{3} t^{\frac{3}{2}} + C \implies y^2 = \frac{4}{3} t^{\frac{3}{2}} + K \]
(K = 2C is just a constant!)

To find the constant we use the initial data \( y(1) = 0 \) which gives \( K + \frac{4}{3} = 0 \) so \( K = -\frac{4}{3} \)

Therefore \( y^2 = \frac{4}{3} t^{\frac{3}{2}} - \frac{4}{3} \), and
\[ y = \sqrt{\frac{4}{3} t^{\frac{3}{2}} - \frac{4}{3}} \quad \text{or} \quad y = -\sqrt{\frac{4}{3} t^{\frac{3}{2}} - \frac{4}{3}} \]
1-b). Find the unique solution to the following initial value problem

\[ \frac{1}{x^2} \frac{dy}{dx} = y(y-2), \quad x = 0 \quad y = 3 \]

**Solution:**

we have again a separable equation. Before doing integration recall from calculus (partial fractions) that

\[ \frac{1}{y(y-2)} = \frac{1}{2} \frac{1}{y-2} - \frac{1}{2} \frac{1}{y} \]

separating the variable and integration gives us:

\[ \frac{1}{y(y-2)} = x^2 \, dx \Rightarrow \int \left( \frac{1}{2} \frac{1}{y-2} - \frac{1}{2} \frac{1}{y} \right) dy = \int x^2 \, dx \]

\[ \Rightarrow \frac{1}{2} \ln |y-2| - \frac{1}{2} \ln |y| = \frac{x^3}{3} + C \Rightarrow \frac{1}{2} \ln \left| \frac{y-2}{y} \right| = \frac{x^3}{3} + C \]

\[ \Rightarrow \ln \left| \frac{y-2}{y} \right| = \frac{2}{3} x^3 + 2C \Rightarrow \frac{y-2}{y} = e^{2C e^{\frac{2}{3} x^3}} \]

\[ \frac{y-2}{y} = Ke^{\frac{2}{3} x^3} \]. It follows by the initial data that \( K = \frac{1}{3} \), hence\[ \frac{y-2}{y} = \frac{1}{3} e^{\frac{2}{3} x^3} \]

\[ y = \frac{2}{1 - \frac{1}{3} e^{\frac{2}{3} x^3}} \]
1. (c) Find the homogeneous solution together with a particular solution, and the general solution to the following linear equation. Also indicate asymptotically what happens to all solutions when $x \to \infty$.

\[ \frac{dy}{dx} + y = x + 2 \]

**Solution:**

*Homogeneous equation:

\[ \frac{dy}{dx} + y = 0 \implies \frac{dy}{dx} = -y \]

\[ y_h = Ke^{-x} \quad \text{(solution to hom. Eq)} \]

*Find a particular solution:

\[ \frac{dy}{dx} + y = x + 2 \]

We shall try \( y_p = Ax + B \)

\[ \frac{dy_p}{dx} + y_p = A + Ax + B = x + 2 \implies \begin{cases} A = 1 \\ B = 1 \end{cases} \]

Hence \( y_p = x + 1 \)

\[ y_G = y_h + y_p = Ke^{-x} + x + 1 \]

General solution
\[ y_G = Ke^{-x} + x + 1 \]

Note that \( \lim_{x \to \infty} y_G = \infty \)

Moreover, \( y_G \sim x + 1 \) as \( x \to \infty \)

i.e. \( \lim_{x \to \infty} y_G - (x + 1) = 0 \)
2. Consider the following modified logistic growth model for a population $P(t)$ measured in biomass units, where $t$ is measured in years.

$$\frac{dP}{dt} = 0.2P(1 - \frac{P}{100}) - 1.5 \left(\text{biomass units per year}\right)$$

a) Find all equilibrium points in the system, and classify them as to stability type.

We have:

$$0.2P - 0.002P^2 - 1.5 = 0$$

Solving for $P^*$, we get:

$$P^* = \frac{-0.2 \pm \sqrt{(0.2)^2 - 4(-0.002)(-1.5)}}{2(-0.002)} = \frac{-0.2 \pm \sqrt{0.04 - 0.012}}{-0.004}$$

$$P_1 \approx 8.16679$$

$$P_2 \approx 91.833$$

$P_1$ is unstable.

$P_2$ is stable.

b) Draw a phase line (horizontally) for this system indicating the information you have found in part a).

![Phase line diagram]

c) On the same pair of axes, sketch the graphs of the solutions of the intitial value problems with initial condition $P(0) = 40$, and $P(0) = 10$ (you do not have to solve the DE to get this information).

![Graphs of solutions]

solutions approach $P_2$ as $t \to \infty$. 
3. Suppose that a 100L tank is initially full of pure water. Starting at time $t=0$ salt water solution is added to the tank at a rate of 2 L/min with concentration of 2.5 gm/L. The mixture in the tank is then drawn off at a rate of 4L/min.

a) Find a differential equation for the amount of salt in solution $S(t)$ (gm). The units for the rate of change of $S(t)$ should be gm/min.

$$V(t) = 100 - 2t \ (L)$$

$$\frac{dS}{dt} (\text{gm/min}) = (2)(2.5)\left(\text{gm/L}\right) - (4)(\text{L/min}) \cdot \frac{S}{100-2t}$$

$$= S - \frac{4S}{100-2t}$$

b) Solving the differential equation from part a), find the amount of salt in solution at time 30 min.

$$\frac{dS}{dt} + \frac{4S}{100-2t} = S$$

Integrating factor $\mu = \exp\left(\int \frac{4}{100-2t} \ dt\right) = \exp\left(-2 \ln |100-2t|\right)$

$= (100-2t)^{-2}$ multiply this through Dill. 89.

$$\frac{dS}{dt}\left(\frac{S}{(100-2t)^2}\right) + \frac{4S}{(100-2t)^3} = \frac{S}{(100-2t)^2}$$

$$\frac{d}{dt}\left[\frac{S}{(100-2t)^2}\right] = \frac{S}{(100-2t)^2}$$

$$\frac{S}{(100-2t)^2} = \int \frac{5}{(100-2t)^2} \ dt = -\frac{5}{2} (100-2t) + C$$

$\Rightarrow$ over
Substitute at \( t=0, \ S=0 \).

\[ 0 = \left(-\frac{5}{2}\right) \frac{1}{100} + C \]

\[ C = \frac{5}{200}. \]

\[ S(t) = \frac{5}{200} (100-2t)^2 - \frac{5}{2} (100-2t) \]

\[ S(30) (gm) = \frac{5}{200} (40)^2 - \frac{5}{2} (40). \]

(3 marks to complete)
4. A radioactive isotope \( N \) decays at a rate of \( 0.1N \) (gm/year). If half of the initial amount of \( N \) decays in 3 years, find the amount of \( N \) present after 10 years.

\[
\frac{dN}{dt} = -\lambda N
\]

\[
N(t) = N_0 e^{-\lambda t}
\]

\[
= N_0 \left( \frac{1}{2} \right)^{t/3}
\]

\[
= N_0 e^{-\frac{\ln 2}{3} t}
\]

\[
\lambda = -\frac{\ln 2}{3}
\]

\[
= -0.2310
\]

\[
N(10) = N_0 e^{-0.2310 (10)}
\]

\[
= N_0 e^{-2.310}
\]

\[
= 0.099 N_0
\]

\[
i.e. \ 9.9\% \ of \ initial
\]

\[
\frac{dN}{dt} = -0.1 N
\]

\[
N(t) = N_0 e^{-0.1 t}
\]

\[
N(10) = N_0 e^{-1}
\]

\[
= N_0 (0.3679)
\]

\[
\frac{1}{2} = 36.79\% \ of \ initial
\]

\[
N(10) = N_0 \left( \frac{1}{2} \right)^{10/3}
\]

\[
= N_0 \left( 0.09923 \right)
\]

\[
9.9\% \ of \ original
\]
5.a) Show directly that any linear combination of \( \cos(2t) \) and \( \sin(2t) \) is a solution of the second order linear equation

\[
\frac{d^2x}{dt^2} = -4x
\]

\[
\frac{d^2}{dt^2} \left( C_1 \cos(2t) + C_2 \sin(2t) \right) = -4 C_1 \cos(2t) - 4 C_2 \sin(2t)
\]

\[
= -4 \left( C_1 \cos(2t) + C_2 \sin(2t) \right)
\]

(4)

b) Find the unique solution to the initial value problem for the equivalent system of first order equations

\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -4x, \quad x(0) = 1, y(0) = -1
\]

(6)

\[\text{First order system is equivalent to} \]

\[\frac{d^2x}{dt^2} = -4x\]

\[x(t) = C_1 \cos(2t) + C_2 \sin(2t)\]

\[y(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t)\]

\[x(0) = C_1 = 1 \quad \Rightarrow \quad C_1 = 1\]

\[y(0) = 2C_2 = -1 \quad \Rightarrow \quad C_2 = -\frac{1}{2}\]

\[
\begin{pmatrix}
  x(t) \\
  y(t)
\end{pmatrix}
= \begin{pmatrix}
  \cos(2t) - \frac{1}{2} \sin(2t) \\
  -2\sin(2t) - \cos(2t)
\end{pmatrix}
\]