1. **Solution I:** The circle of radius \( r > 0 \) going once around \( z_0 \) counterclockwise can be parametrized by \( \gamma_r(t) = r \cdot \exp(it) + z_0 \) with \( t \in [0, 2\pi] \). We then have

\[
\int_{\gamma_r} \frac{1}{z - z_0} \, dz = \int_0^{2\pi} \frac{1}{\gamma_r(t) - z_0} \gamma_r'(t) \, dt = \int_0^{2\pi} \frac{1}{(r \cdot \exp(it) + z_0) - z_0} \cdot (ri \exp(it)) \, dt
\]

\[
= \int_0^{2\pi} \frac{1}{r \cdot \exp(it)} \cdot ri \exp(it) \, dt = \int_0^{2\pi} i \, dt = 2\pi i.
\]

Notice that the answer does not depend on \( r \), which is now something we expect. The function \( \frac{1}{z - z_0} \) is differentiable on \( \mathbb{C} \setminus \{z_0\} \), and for any two real numbers \( r_0 \) and \( r_1 \), \( \gamma_{r_0} \) is homotopic to \( \gamma_{r_1} \) in \( \mathbb{C} \setminus \{z_0\} \), and therefore

\[
\int_{\gamma_{r_0}} \frac{1}{z - z_0} \, dz = \int_{\gamma_{r_1}} \frac{1}{z - z_0} \, dz
\]

by Cauchy’s theorem.

**Solution II:** Another way to write the previous parameterization is \( \gamma_r(t) = r \cos(t) + ir \sin(t) + z_0 \), again with \( t \in [0, 2\pi] \). Writing the parameterization this way we have

\[
\int_{\gamma_r} \frac{1}{z - z_0} \, dz = \int_0^{2\pi} \frac{1}{\gamma_r(t) - z_0} \gamma_r'(t) \, dt
\]

\[
= \int_0^{2\pi} \frac{1}{(r \cos(t) + ir \sin(t) + z_0) - z_0} \cdot (-r \sin(t) + ir \cos(t)) \, dt
\]

\[
= \int_0^{2\pi} \frac{1}{r \cos(t) + ir \sin(t)} \cdot (-r \sin(t) + ir \cos(t)) \, dt
\]

\[
= \int_0^{2\pi} i(r \cos(t) + ir \sin(t)) \left( \frac{1}{r \cos(t) + ir \sin(t)} \right) \, dt = \int_0^{2\pi} i \, dt = 2\pi i.
\]

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2. Let \( f(z) = 3|z|^2 \), and \( z = x + iy \), with \( x, y \in \mathbb{R} \).

(a) The curve \( \gamma_1 \) can be parameterized by \( \gamma_1(t) = (x + iy) \cdot t \) with \( t \in [0, 1] \). We then have

\[
\int_{\gamma_1} 3|z|^2 \, dz = \int_0^1 3|\gamma_1(t)|^2 \gamma'_1(t) \, dt = \int_0^1 3t^2(x^2 + y^2) \cdot (x + iy) \, dt
\]

\[
= \int_0^1 3t^2 \left((x^3 + xy^2) + i(x^2y + y^3)\right) \, dt
\]

\[
= t^3 \left((x^3 + xy^2) + i(x^2y + y^3)\right) \Big|_{t=0}^{t=1} = (x^3 + xy^2) + i(x^2y + y^3).
\]

(b) To parameterize the curve \( \gamma_2 \) we parameterize the two pieces separately. The first piece, \( \gamma_{2,1} \) can be parameterized by \( \gamma_{2,1}(t) = xt \), for \( t \in [0, 1] \). This piece of the integral then becomes

\[
\int_{\gamma_{2,1}} 3|z|^2 \, dz = \int_0^1 3|\gamma_{2,1}(t)|^2 \gamma'_{2,1}(t) \, dt = \int_0^1 3t^2x^2 \cdot x \, dt
\]

\[
= \int_0^1 3t^2x^3 \, dt = t^3x^3 \Big|_{t=0}^{t=1} = x^3.
\]

The second leg of the integral can be parameterized by \( \gamma_{2,2}(t) = x + ity \), \( t \in [0, 1] \), and the integral on this piece is

\[
\int_{\gamma_{2,2}} 3|z|^2 \, dz = \int_0^1 3|\gamma_{2,2}(t)|^2 \gamma'_{2,2}(t) \, dt = \int_0^1 3(x^2 + t^2y^2) \cdot iy \, dt
\]

\[
= \int_0^1 3i(x^2y + t^2y^3) \, dt = i(3x^2yt + t^3y^3) \Big|_{t=0}^{t=1} = i(3x^2y + y^3).
\]

Adding the two pieces together we get

\[
\int_{\gamma_2} 3|z|^2 \, dz = \int_{\gamma_{2,1}} 3|z|^2 \, dz + \int_{\gamma_{2,2}} 3|z|^2 \, dz = x^3 + i(3x^2y + y^3).
\]

(c) The function from part (a) is \( F_1(x + iy) = (x^3 + xy^2) + i(x^2y + y^3) \), i.e., the real and imaginary parts of \( F_1 \) are \( u_1(x, y) = x^3 + xy^2 \) and \( v_1(x, y) = x^2y + y^3 \). Both functions are clearly infinitely differentiable as real functions of two variables. To
see if $F_1$ is (complex) differentiable, we therefore just need to check the Cauchy-Riemann equations.

The only point at which the Cauchy-Riemann equations are satisfied is $(x, y) = (0, 0)$, i.e., $F_1$ is differentiable only at $(0, 0)$, and is analytic (holomorphic) nowhere.

\[
\begin{align*}
\frac{\partial u_1}{\partial x} &= 3x^2 + y^2 & \frac{\partial u_1}{\partial y} &= 2xy \\
\frac{\partial v_1}{\partial x} &= 2xy & \frac{\partial v_1}{\partial y} &= x^2 + 3y^2 
\end{align*}
\]

(d) The function from part (b) is $F_2(x + iy) = x^3 + i(3x^2y + y^3)$, i.e., $F_2$ has real and imaginary parts $u_2(x, y) = x^3$ and $v_2(x, y) = 3x^2y + y^3$. The Cauchy-Riemann equations are satisfied only when $y = 0$, i.e., $F_2$ is differentiable only on the real axis, and is analytic (holomorphic) nowhere.

\[
\begin{align*}
\frac{\partial u_2}{\partial x} &= 3x^2 & \frac{\partial u_2}{\partial y} &= 0 \\
\frac{\partial v_2}{\partial x} &= 6xy & \frac{\partial v_2}{\partial y} &= 3x^2 + 3y^2 
\end{align*}
\]

**Note:** In class we proved that a function $f$ has a complex antiderivative on a domain $D$ if and only if path-integrals in $D$ only depend on the endpoints. The proof that path-independence implied the existence of an anti-derivative involved integrating along a path, and then proving that the resulting function was differentiable with derivative $f$. This question deals with two aspects of that argument. First, since the answers for (a) and (b) are different, the integral of $3|z|^2$ clearly depends on the path (and so by the theorem, $3|z|^2$ has no complex antiderivative). Second, even though we can still construct functions from $f$ by integrating (e.g. $F_1$ and $F_2$), these functions are not necessarily complex differentiable. Note that this is a departure from the usual behaviour in real one-variable calculus, where the fundamental theorem of calculus shows that the integral of any continuous function is differentiable.

3. In this problem we use the estimate from class: If $\gamma$ is any contour, $f(z)$ a function and $M$ a real number such that $|f(z)| \leq M$ for all $z \in \gamma$, then

$$\left| \int_\gamma f(z) \, dz \right| \leq M \cdot \text{length(}\gamma\text{)}.$$
(a) The contour $\gamma$ is a circle of radius 3 and therefore has length $2\pi$. To establish the estimate we just need to show that $\frac{1}{|z^2 - i|} \leq \frac{1}{8}$ on $\gamma$. Here are two equivalent ways to show this:

**Geometric:** If $z$ is on the circle $|z| = 3$ then $z^2$ is on the circle of radius $|z|^2 = 9$.

The closest that a point on the circle $|w| = 9$ gets to $i$ is 8, and the farthest that a point on the circle is from $i$ is 10, and this gives the inequalities

$$8 \leq |z^2 - i| \leq 10.$$ 

Taking the reciprocal yields $\frac{1}{10} \leq \left|\frac{1}{z^2 - i}\right| \leq \frac{1}{8}$.

**Triangle Inequalities:** By the triangle inequality we have

$$|z^2 - i| \geq |z^2| - |i| = |z|^2 - 1 = 9 - 1 = 8.$$ 

Taking the reciprocal again gives the inequality $\left|\frac{1}{z^2 - i}\right| \leq \frac{1}{8}$.

Given this bound (by either method) on $\left|\frac{1}{z^2 - i}\right|$ along $\gamma$, we obtain the estimate

$$\left|\int_\gamma \frac{dz}{z^2 - i}\right| \leq \frac{1}{8} \cdot 6\pi = \frac{3\pi}{4}.$$ 

(b) Recall that $\text{Log}(z) = \ln |z| + i \text{Arg}(z)$. In particular, if $z$ is on the unit circle then $\text{Log}(z) = \ln |z| + i \text{Arg}(z) = \ln 1 + i \text{Arg}(z) = i \text{Arg}(z)$, and so $|\text{Log}(z)| = |\text{Arg}(z)|$. By the definition of $\gamma$ we have $0 \leq \text{Arg}(z) \leq \frac{\pi}{2}$. Since the length of $\gamma$ is also $\frac{\pi}{2}$, we obtain the estimate

$$\left|\int_\gamma \text{Log}(z)\,dz\right| \leq \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}.$$ 

(c) In class we saw that for any complex number $w$, $|\exp(w)| = e^{\text{Re}(w)}$, and hence $|\exp(\sin(z))| = e^{\text{Re}(\sin(z))}$. By question 4(a) from Homework 4, $\text{Re}(\sin(z)) = \sin(x) \cosh(y)$. The curve $\gamma$ lies on the imaginary axis, and so $x = 0$ along $\gamma$, and in particular

$$|e^{\sin(z)}| = e^{\sin(0) \cosh(y)} = e^0 = 1$$ 

along $\gamma$. Since $\gamma$ has length 1, this gives the estimate $\left|\int_\gamma e^{\sin(z)}\,dz\right| \leq 1 \cdot 1 = 1$. 

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(d) Put \( \gamma : |z| = 3 \). We have
\[
\left| \int_\gamma \frac{\log(z)}{z - 4i} \right| \leq \frac{|\ln|z|| + |\text{Arg}(z)|}{||z| - 4i||}
\]
so that
\[
\max_{z \in \gamma} \left| \frac{\log(z)}{z - 4i} \right| \leq \frac{\ln(3) + \pi}{|3 - 4|} = \ln(3) + \pi.
\]
Now the length is:
\[
L = (2\pi)3 = 6\pi
\]
so that
\[
\left| \int_\gamma \frac{\log(z)}{z - 4i} \, dz \right| \leq 6\pi(\pi + \ln(3)).
\]

4.

(a) Let \( \gamma \) is the quarter-circle centered at the origin and extending from 2 to \( 2i \).
Parametrize \( \gamma \) and compute
\[
\int_\gamma (z^2 - 3|z| + \text{Im}(z)) \, dz.
\]
We put \( \gamma(t) = 2e^{it}, \, 0 \leq t \leq \pi/2 \). Then \( \gamma'(t) = 2ie^{it} \) and
\[
\int_\gamma (z^2 - 3|z| + \text{Im}(z)) \, dz = \int_0^{\pi/2} (4e^{2it} - 6 + 2\sin(t)) \, 2ie^{it} \, dt
\]
\[
= \left[ \frac{8}{3}e^{3it} - 12e^{it} + \frac{1}{i}e^{2it} - 2t \right]_0^{\pi/2} = \frac{28}{3} - \pi - \frac{38}{3}i.
\]

(b) Let \( f(z) = z \). Then \( f \) is continuous on a directed smooth curve \( \gamma \). It has an antiderivative \( F(z) = \frac{z^2}{2} \). If \( z = z(t), \, a \leq t \leq b \) is an admissible parametrization of \( \gamma \), then we have
\[
\int_\gamma z \, dz = \int_a^b z(t) \, z'(t) \, dt = \int_\alpha^\beta z(t) \, z'(t) \, dt
\]
\[
= F(\beta) - F(\alpha) = \frac{\beta^2 - \alpha^2}{2}.
\]
Here \( \alpha = \gamma(a) \) and \( \beta = \gamma(b) \).
(c) If $\gamma$ is a closed contour, by (2) above,
\[ \int_{\gamma} z \, dz = 0. \]

(d) The almost the same argument as for (2) above works for $f(z) = z^n$ with $n = 0, 1, 2, \cdots$. Then $f$ is continuous on a directed smooth curve $\gamma$. It has an antiderivative $F(z) = \frac{z^{n+1}}{n+1}$. If $z = z(t)$, $a \leq t \leq b$ is an admissible parametrization of $\gamma$, then we have
\[ \int_{\gamma} z^n \, dz = \int_{a}^{b} z(t)^n \cdot z'(t) \, dt = [F(z)]_{a}^{b} \]
\[ = F(b) - F(a) = \frac{\beta^{n+1} - \alpha^{n+1}}{n+1}. \]
But $\gamma(a) = \gamma(b)$ as $\gamma$ is closed. Hence
\[ \int_{\gamma} z^n \, dz = 0 \quad \text{for } n = 0, 1, 2, \cdots. \]

(e) Let $\gamma_1$ be the line segment from 1 to $|\alpha|$ along the real axis, and $\gamma_2$ be a circular arc centered at the origina and of radius $|\alpha|$ which extends from $|\alpha|$ to $\alpha$. The union $\gamma_1 + \gamma_2 - \gamma$ forms a closed contour. Since the integrand $1/z$ is analytic everywhere inside $D$, by the Cauchy integral formula, we have
\[ \int_{\gamma} \frac{dz}{z} = \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z}. \]
Now write $\alpha = |\alpha|e^{i\text{Arg}(\alpha)} \text{Arg}(\alpha) \in (-\pi, \pi)$. Then
\[ \int_{\gamma_1} \frac{dz}{z} = \int_{1}^{\alpha} \frac{dt}{t} = \ln|\alpha| \]
\[ \int_{\gamma_2} \frac{dz}{z} = \int_{0}^{\text{Arg}(\alpha)} \frac{i\cdot e^{i\theta}}{r} \cdot e^{i\theta} \, d\theta = i\text{Arg}(\alpha). \]
Combining the results we have
\[ \int_{\gamma} \frac{dz}{z} = \ln|\alpha| + i\text{Arg}(\alpha) = \log(\alpha). \]